



On Jackson's Theorem

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ABSTRACT

We prove that for a function $f \in W_p^1[-1,1]$, $0 < p < 1$ and n,r in N , we have:

$$\left(\int_{-1}^1 f(x)dx - \sum_{j=1}^n \omega_j f(x_j) \right) \leq c(r)n^{-1} \int_0^{1/n} \frac{\omega_\varphi^{r-1}(f',u)_p}{u^2} du$$

where $-1 < x_1 < \dots < x_n < 1$ the roots of Legendre polynomial, and $\omega_\varphi^m(g,\delta)_p$, is the Ditzian-Totik m th modulus of smoothness of g in L_p .

Keywords: Approximation of functions, Best approximation, Degree of approximation.

1. INTRODUCTION

Let L_p , $0 < p < \infty$ be the set of all functions, which are measurable on $[a,b]$, such that:

$$\|f\|_{L_p[a,b]} := \left(\int_a^b |f(x)|^p dx \right)^{1/p} < \infty.$$

And let $W_p^r[a,b]$ be the space of functions that

$f^{(r)} \in L_p[a,b]$ and $f^{(r-1)}$ is absolutely continuous in $[a,b]$.

We believe that for approximation in $L_p, p < 1$ the measure of smoothness $\omega_\varphi^r(f,\delta)_p$ introduced by Ditzian and Totik [1] is the appropriate tool. Recall that

$$\omega_\varphi^r(f,\delta,[a,b])_p = \sup_{0 < h \leq \delta} \left(\int_a^b \left| \Delta_{h\varphi(x)}^r(f,x,[a,b]) \right|^p dx \right)^{1/p},$$

where

$$\Delta_{h\varphi(x)}^r(f,x,[a,b]) := \begin{cases} \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} f\left(x - \frac{rh}{2} + kh\right), & \text{if } x \pm \frac{rh}{2} \in [a,b] \\ 0, & \text{o.w.} \end{cases}$$

For $[a,b] := [-1,1]$ for simplicity we write

$$\|f\|_p = \|\cdot\|_{L_p[a,b]}, \text{ and } \omega_\varphi^r(f,\delta)_p := \omega_\varphi^r(f,\delta,[a,b])_p.$$

Recall that the rate of best n th degree polynomial approximation is given by

$$E_n(f)_p := \inf_{p_n \in \Pi_n} \|f - p_n\|_p$$

where Π_n denote the set of all algebraic polynomials of degree not exceeding n .

To prove our theorem we need the following direct result given by:

Theorem 1.1: For n,r in N and $f \in L_p[-1,1]$

$$E_n(f)_p \leq c \omega_\varphi^r(f,n^{-1})_p \tag{1}$$

where c is a constant depending on r and p (if $p < 1$). For $1 \leq p \leq \infty$ (1) was proved by Ditzian and Totik [1] and for $0 < p < 1$, it has been proved by DeVore, Leviatan and Yu [2].

Now, consider the Gaussian Quadrature process [3]

$$\int_{-1}^1 f(x)dx \approx \sum_{j=1}^n \omega_j f(x_j) =: I_n(f) \tag{2}$$

based on the roots $-1 < x_1 < \dots < x_n < 1$ of the n th Legendre polynomial. Since this exact polynomial of degree less than $2n$, we get for the error

$$e_n(f) = \int_{-1}^1 f(x)dx - I_n(f)$$

in (2) by the definition of the degree of best approximation we have

$$e_n(f) \leq 2E_{2n-1}(f)_\infty \tag{3}$$

where

$$\|f\|_\infty := \sup_{x \in [-1,1]} |f(x)|$$



(note that $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j$). The crude method of estimating $e_n(f)$ consists of applying Jackson estimate on the right of (3), from (1) we get the sharp inequality

$$e_n(f) \leq c \omega_\varphi^r(f, n^{-1})_\infty \quad (4)$$

which already takes in to account the possibly less smooth behavior of f at ± 1 . However the supremum norm in (5) is still too rough, and the natural question is whether for smooth functions one can get upper bounds for $e_n(f)$ using certain $L_p, p < 1$ quasi-norm.

R. A. DeVore and L. R. Scott [3] found such estimates, they proved:

$$e_n(f) \leq c(s)n^{-s} \int_{-1}^1 |f^{(s)}(x)| (1-x^2)^{s/2} dx \quad (5)$$

First for $s=1$ which obviously implies:

$$e_n(f) \leq cn^{-1} E_{2n-2}(f')_{\varphi, p} \quad p \geq 1 \quad (6)$$

where $E_n(f)_{\varphi, p}$ means the best weighted approximation with weight $\varphi(x)$ of f in L_p defined by:

$$E_n(f)_{\varphi, p} := \inf_{p_n \in \Pi_n} \|\varphi(f - p_n)\|_p.$$

They then proceeded to estimate $E_n(f')_p, p \geq 1$, using higher derivatives of f which finally yielded (5) for any $s \geq 1$.

2. THE MAIN RESULT

Using (6) we obtain the following theorem

Theorem 2.1: For $f \in W_p^1[-1,1], 0 < p < 1$ we have:

$$e_n(f) \leq c(r)n^{-1} \int_0^1 \frac{\omega_\varphi^{r-1}(f', u)_p}{u^2} du \quad (7)$$

Of course the convergence of the integral on the right implies that f is L_p equivalent of a locally absolutely continuous function. We use this equivalent representative of f in the quadrature formula (Otherwise, we don't have even $e_n(f) = o(1)$)

Proof: Let $p_n \in \Pi_n$ be the best approximating polynomial for f in $L_p[-1,1], p < 1$. Then:

$f = p_n + \sum_{k=0}^{\infty} (p_{2^{k+1}n} - p_{2^k n})$ in $L_p[-1,1]$ (i.e. the expression in the right is the L_p equivalent of f which we need). From (6) and Markov-Bernstein type inequality (see for example [4])

$$\begin{aligned} e_n(f) &\leq cn^{-1} E_{2n-2}(f')_{q, \varphi} \quad q \geq 1 \\ &\leq cn^{-1} E_n(f')_{q, \varphi} \\ &\leq cn^{-1} \|\varphi(f' - p'_n)\|_q \\ &\leq cn^{-1} \sum_{k=0}^{\infty} 2^{k+1} n \|\varphi(p_{2^{k+1}n} - p_{2^k n})\|_q. \end{aligned}$$

Then using the fact that any two quasi norms are equivalent on the space of algebraic polynomials of a fixed degree we have:

$$e_n(f) \leq c(p) \sum_{k=0}^{\infty} 2^{k+1} n E_{2^k n}(f)_p \quad p < 1.$$

In view of (1) we get

$$e_n(f) \leq c(p) \sum_{k=0}^{\infty} 2^k n \omega_\varphi^r(f, 2^{-k} n^{-1})_p.$$

Now since $f \in W_p^1[-1,1], 0 < p < 1$, so that :

$$\begin{aligned} e_n(f) &\leq c(p)n^{-1} \sum_{k=0}^{\infty} 2^k n \omega_\varphi^r(f, 2^{-k} n^{-1})_p \\ &\leq c(p)n^{-1} \int_0^1 \frac{\omega_\varphi^{r-1}(f', u)_p}{u^2} du. \end{aligned}$$

Provided the last integral convergence ♦

As a final remark, we mention that similar bounds hold for many other systems of nodes and in (7) the right hand side has the order

$$\left(\int_{-1}^{x_n} |f|^p + \int_{x_n}^1 |f|^p \right)^{1/p}, \text{ for any } f \text{ constructed from analytic}$$

functions, $|x \pm 1|^s$ and iterated logarithms of these, which means that (7) is the best possible estimate for such functions.

**REFERENCES**

- [1] Z. Ditzian and V. Totik, Moduli of smoothness, Springer-Verlag, Berlin, 1987.
- [2] R. A. DeVore, D. Leviatan and X. M. Yu, Polynomial approximation in $L_p(0 < p < 1)$, Constr.approx., **8**, (1992), 187-201.
- [3] R. A. DeVore, and L. R. Scott, Error bounds for Gaussian quadrature and weighted L^1 polynomial approximation, SIAM, J. Numer. Anal, **2**, (1984), 400-412.
- [4] K. Kopotun and A. Shadrin , On k-monotone approximation by free knots splines, Siam J. Math. Anal., **34**, (2003), 901-924.