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Weak Forms of Regular Spaces

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ABSTRACT

In this paper we introduce some definitions of weak types of regular spaces using the weak ω −open set defined in [4], and prove some relations among them. Then we prove some theorems related to them.

Keywords: *Weak open sets, weak regular spaces, ω −open set.*

1. INTRODUCTION

In this article let us prepare the background of the subject. Throughout this paper , (X,T) stands for topological space. Let A be a subset of X. A point x in X is called *condensation* point of A if for each U in T with x in U, the set $U \cap A$ is uncountable [1]. In 1982 the ω −closed set was first introduced by H. Z. Hdeib in [1], and he defined it as: A is ω −*closed* if it contains all its condensation points and the ω −*open* set is the complement of the ω −closed set. It is not hard to prove: any open set is ω −open. For a subset A of X, the $\omega-$ *interior* of the set A defined as the union of all $\omega-$ open sets contained in A, and denoted by $int_\omega(A)$. By $int(A)$, we will denoted to the *interior* of the set A. Also we would like to say that the collection of all ω −open subsets of X forms topology on X. The closure of A will be denoted by $cl(A)$, while the intersection of all ω −closed sets in X which containing A is called the ω −*closure* of A, and will denote by $cl_\omega(A)$. Note that $cl_\omega(A) \subset cl(A)$.

Definition 1.1: A subset A of a space X is called An ω −*set*, [4] if $A = U \cap V$, where U is an open set and $int(V) = int_\omega(V)$. And we say that X satisfy the ω −*condition*, [3] if every ω −open set is ω −set.

Definition 1.2: [5] Let X be a topological space. We say that a subset A of X is ω −*compact*, if for each cover of ω −open sets from X contains a finite subcover for A.

For our main results we also need the following definitions:

Definition 1.3: Let X be a topological space. For each $x \neq y \in X$, there exists a set U, such that $x \in U, y \notin U$, and there exists a set V such that $y \in V, x \notin V$, then X is called

1. $\omega - T_1$ *space*, [3] if U is open and V is ω −open sets in X.
2. $\omega^\star - T_1$ *space*, [5] if U and V are ω −open sets in X.

Definition 1.4: [3] Let X be a topological space. And for each $x \neq y \in X$, there exist two disjoint sets U and V with $x \in U$ and $y \in V$, then X is called:

1. $\omega - T_2$. *space* if U is open and V is ω −open sets in X.
2. $\omega^\star - T_2$ *space* if U and V are ω −open sets in X.

2. FORMS OF REGULAR SPACES

In this section let us introduce our new definition and it's related theorems.

Definition 2.1: Let X be a topological space. If for a given $x \in X$ and a set $F \subset X$, with $x \notin F$, there exist two disjoint sets U and V with $x \in U$ and $F \subset V$ then X named as the following F, U and V:

Type of regularity	*F*	*U*	*V*
regular [2]	closed	open	open
ω $-$ ***regular***[3]	closed	ω −open	open
$\omega_1 - $ ***regular***	closed	open	ω −open
$\omega_2 - $ ***regular***	closed	ω −open	ω −open
$\omega_3 - $ ***regular***	ω −closed	open	open
$\omega_4 - $ ***regular***	ω −closed	ω −open	open
$\omega_5 - $ ***regular***	ω −closed	open	ω −open
$\omega^\star -$ ***regular***[3]	ω −closed	ω −open	ω −open

Thus let us introduce the following definition:

Definition 2.2: For any topological space.

1. *regular*, T_1 space is called T_3 *space*.[2]
2. An $\omega -$ *regular*, $\omega - T_1$ space is called $\omega - T_3$ *space*.[3]
3. An $\omega_1 -$ *regular*, $\omega - T_1$ space is called $\omega_1 - T_3$ *space*.
4. An $\omega_2 -$ *regular* , $\omega^\star - T_1$ space is called $\omega_2 - T_3$ *space*.
5. An $\omega_3 -$ *regular* , T_1 space is called $\omega_3 - T_3$ *space*.
6. An $\omega_4 -$ *regular* , $\omega - T_1$ space is called $\omega_4 - T_3$ *space*.
7. An $\omega_5 -$ *regular* , $\omega - T_1$ space is called $\omega_5 - T_3$ *space*.
8. An $\omega^\star -$ *regular* , $\omega^\star - T_1$ space is called $\omega^\star - T_3$ *space*. [3]

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First of all we shall study the relationships among the definitions of weak regular spaces above:

Theorem 2.3: If X is $regular$ space, then it is $-$ $regular$.

Proof: Let X be a $regular$ space, x is a point in X, and F be a closed set not containing x. Then since X is $regular$ space, and any open set is also ω −open, so there exist an ω −open set U containing x and an open set V containing F. This implies that X is $\omega - regular$ ✤

Remark 2.4: The converse of Theorem 2.3 is true if X satisfies the ω −condition, as we see in the following theorem:

Theorem 2.5: If X is an $\omega - regular$ space satisfies the ω −condition, then it is $regular$.

Proof: Let X be an $\omega - regular$ space, x is a point in X, and F be a closed set not containing x. Then by our hypothesis we can find two sets V open containing F, and U is ω −open containing x, by Definition 1.1 and Proposition 3.11 of [4] we can get U is open. Thus X is $regular$ ✤

Theorem 2.6: Let X be a topological space satisfies the ω −condition. X is $\omega - regular$ space if and only if it is $\omega_1 - regular$.

Proof: Let X be an $\omega - regular$ space, x is a point in X, and F be a closed set not containing x. Then there exist two sets U is ω −open (by the ω −condition it becomes open) set containing x. And V is open (also it is ω −open) set containing F. Hence X is $\omega_1 - regular$ space.

Considering opposite side, let X be an $\omega_1 - regular$ space, x is a point in X, and F be a closed set not containing x. Then there exists an open set U, so it is ω −open and containing x. Also there is an ω −open set V (so it is open by the ω −condition) which containing F.

Therefore X is $\omega - regular$ space ✤

Using the fact that any open set is ω −open and Definition 2.1 we can prove the following theorems:

Theorem 2.7: If X is an $\omega_1 - regular$ space, then it is $\omega_2 - regular$.

Theorem 2.8: If X is an $\omega_2 - regular$ space satisfies the ω −condition, then it is $\omega_1 - regular$.

Theorem 2.9: If X is $\omega_3 - regular$ space, then it is $\omega_2 - regular$.

Proof: Let X be an $\omega_3 - regular$ space, x is a point in X, and F be a closed set not containing x. Then since the collection of all closed subsets of X is contained in the collection of all ω −closed subsets of X, so that there is an open set (hence ω −open) containing x, and also there is an open set (hence ω −open) containing F. Therefore X is $\omega_2 - regular$ space ✤

Theorem 2.10: If X is an $\omega_2 - regular$ space satisfies the ω −condition, then it is $\omega_3 - regular$.

Proof: Let X be an $\omega_2 - regular$ space, x is a point in X, and F be na ω −closed set not containing x. Then since X satisfy the ω −condition so that the collection of all closed subsets of X and the collection of all ω − closed subsets of X are the same. This implies that F is also a closed set. Since X is $\omega_2 - regular$ space there is an ω −open (hence open) set containing x, and also there is an ω −open (hence open) set containing F. Thus X is $\omega_3 - regular$ space ✤

As the same way of the proof of the above theorems we can prove the following theorems:

Theorem 2.11: If X is an $\omega_3 - regular$ space, then it is $\omega_4 - regular$.

Theorem 2.12: If X is an $\omega_4 - regular$ space satisfies the ω −condition, then it is $\omega_3 - regular$.

Theorem 2.13: Let X be a topological space satisfies the ω −condition. X is $\omega_4 - regular$ space if and only if it is $\omega_5 - regular$.

Theorem 2.14: Let X be a topological space satisfies the ω −condition. X is $\omega_5 - regular$ space if and only if it is $\omega^{\star} - regular$.

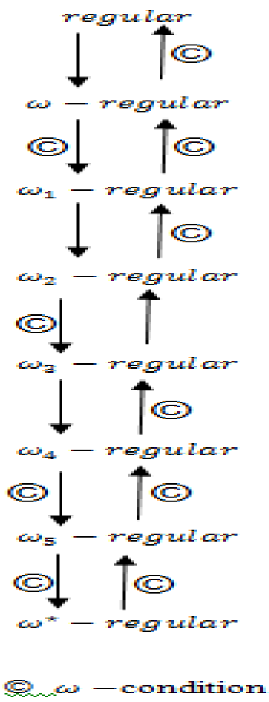

Figure 1: Relations among weak regular spaces

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The above figure summarizes the Theorems from Theorem 2.3 to Theorem 2.14.

Theorem 2.15: For any topological space.

1. Any compact $\omega - T_2$ space, is $\omega_1 - T_3$ space.
2. Any compact $\omega^{\star} - T_2$ space, is $\omega_2 - T_3$ space.
3. Any ω −compact T_2 space, is $\omega_3 - T_3$ space.
4. Any ω −compact $\omega - T_2$space, is $\omega_4 - T_3$ space.
5. Any ω −compact $\omega - T_2$space, is $\omega_5 - T_3$ space.

Proof of (5): Let (X, T) be an ω −compact $\omega - T_2$ space. Then (X, T) is $\omega - T_1$ space. It is suffices to prove (X, T) is ω_4 −regular. Let F be an ω −closed subset of X, and y in X but not in F, so $y \in X \backslash F$. Since X is $\omega - T_2$, so for each x in F there are disjoint sets H_x open in X and containing x, and G_x ω −open set in X containing y. $\{H_x, x \in F\}$ is a cover for F consists of open sets. Since F is ω −closed subset of an ω −compact space so by Theorem 1.9.5 in [3], we get that F is ω −compact. Hence it is compact. Therefore there are $x_1, x_2, \dots, x_n \in F$, such that $F \subset \cup_{i=1}^{n} H_{x_i}$. Let $H = \cup_{i=1}^{n} H_{x_i}$ and $G = \cap_{i=1}^{n} G_{x_i}$. Then G is ω −open set containing y and H is open set containing F. G and H are disjoint if G and H are not disjoint so there exists $z \in G \cap H$, then $z \in G_{x_i}$ for each i and $z \in H_{x_i}$ for some i. This contradicts G_{x_i} and H_{x_i} are disjoint.

The proof of the other cases are similar ✜

Theorem 2.16: An $\omega - T_1$ space X is $\omega - T_3$ if and only if for each a in X and each open set U containing a, there is an ω −open set W containing a and its closure contained in U.

Proof: Let X be an $\omega - regular$ space, a is an element in X, and U is an open set containing a. Then $X \backslash U$ is closed set not containing a, so there are disjoint sets W ω −open set containing a, and V is open set containing $X \backslash U$. Since $W \subset X \backslash V$, and $X \backslash V$ is closed set in X, then $cl(W) \subset X \backslash V$. Therefore $cl(W) \subset X \backslash V \subset X \backslash (X \backslash U) = U$.
For the converse. Let a is an element in X, and C be a closed set not containing a. Then $X \backslash C$ is open set containing a, so there exist an ω −open set containing W such that W containing a and $cl(W) \subset X \backslash C$. Then W is an ω −open set containing a and $X \backslash cl(W)$ is ω −open set containing . Thus X is $\omega - regular$. This completes the proof ✜

Theorem 2.17: An $\omega - T_1$ space X is $\omega_1 - T_3$ if and only if for each a in X and each open set U containing a, there is an open set W containing a and its ω −closure contained in U.

Proof: Let X be an $\omega_1 - regular$ space, a is an element in X, and U is an open set containing a. Then $X \backslash U$ is closed set not containing a, so there are disjoint sets W open set containing a, and V is ω −open set containing $X \backslash U$. Then since $W \subset X \backslash V$, and $X \backslash V$ is ω −closed set in X, then $cl_{\omega}(W) \subset X \backslash V \subset U$. Therefore W is the required open set.
For the converse. Let a is an element in X, and C be a closed set not containing a. Then $X \backslash C$ is open set containing a, so there exist an open set W containing a and $X \backslash cl_{\omega}(W) \subset X \backslash C$. Then W is an open set containing a, and $X \backslash cl_{\omega}(W)$ is ω −open set containing C. Thus X is $\omega_1 - regular$. This completes the proof ✜

As the same way we can prove the following theorems:

Theorem 2.18: An $\omega^{\star} - T_1$ space X is $\omega_2 - T_3$ if and only if for each a in X and each open set U containing a, there is an ω −open set W containing a and its ω −closure contained in U.

Theorem 2.19: A T_1 space X is $\omega_3 - T_3$ if and only if for each a in X and each ω −open set U containing a, there is an open set W containing a and its closure contained in U.

Theorem 2.20: An $\omega - T_1$ space X is $\omega_4 - T_3$ if and only if for each a in X and each ω −open set U containing a, there is an ω −open set W containing a and its closure contained in U.

Theorem 2.21: An $\omega - T_1$ space X is $\omega_5 - T_3$ if and only if for each a in X and each ω −open set U containing a, there is an open set W containing a and its ω −closure contained in U.

Theorem 2.22: An $\omega^{\star} - T_1$ space X is $\omega^{\star} - T_3$ if and only if for each a in X and each ω −open set U containing a, there is an ω −open set W containing a and its ω −closure contained in U.

To introduce the next theorem we need the following definitions from [6]:

Definition 2.23: [6] Let Λ be an index set and $\{X_{\lambda}, \lambda \in \Lambda\}$ a collection of sets. The *cartesian product* $X = \prod_{\lambda \in \Lambda} X_{\lambda}$ is the collection of all functions x with domain Λ having the property that the value x_{λ} of x at λ belongs to the set X_{λ}. For $\lambda \in \Lambda$, the function: $p_{\lambda}: X \longrightarrow X_{\lambda}$, defined by $p_{\lambda}(x) = x_{\lambda}$, $x \in X$ is called the *projection map* of X on the λth *coordinate set* X_{λ}.

Definition 2.24: [6] Let Λ be an index set and $\{X_{\lambda}, \lambda \in \Lambda\}$ a collection of nonempty topological spaces. The *product topology* for $X = \prod_{\lambda \in \Lambda} X_{\lambda}$ is the topology generated by the subbasis S of all sets of the form $p_{\lambda}^{-1}(O_{\lambda})$, $\lambda \in \Lambda$, where O_{λ} is open in X_{λ}.

According to Definition 2.24, a basis for the product topology for X consists of all finite intersections $\cap_{i=1}^{n} p_{\lambda_i}^{-1}(O_{\lambda_i}), \lambda_i \in \Lambda, 1 \leq i \leq n$, where each O_{λ_i} is an open subset of X_{λ_i}. [2]

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Theorem 2.25: The product of any collection of $\omega - T_3$ (resp. $\omega_1 - T_3$, $\omega_2 - T_3$, $\omega_3 - T_3, \omega_4 - T_3$, $\omega_5 - T3$ and $\omega\star - T3$) is $\omega - T3$ (resp. $\omega 1 - T3$, $\omega 2 - T3$, $\omega_3 - T_3, \omega_4 - T_3$, $\omega_5 - T_3$ and $\omega^\star - T_3$).

Proof: Let $\{X_\lambda, \lambda \in \Lambda\}$ a collection of $\omega - T_3$ spaces. Suppose $X = \prod_{\lambda\in\Lambda} X_\lambda$. Let a be an element of X, and U an open set containing a. To prove X is $\omega - T_3$ we can use Theorem 2.16, so we must find an ω −open set W in X containing a and its closure is contained in U. Let $\cap_{i=1}^{n} p_{\lambda_i}^{-1}(U_{\lambda_i})$ be an open set in X containing a and subset of U such that each U_{λ_i} is an open set in X containing $p_{\lambda_i}(a)$. Since each X_{λ_i} is $\omega - T_3$, then for each $i = 1,2,\dots,n$ we can find an ω −open set W_{λ_i} in X_{λ_i} satisfying $p_{\lambda_i}(a) \in W_{\lambda_i}$ and $cl(W_{\lambda_i}) \subset U_{\lambda_i}$. Then $W = \cap_{i=1}^{n} p_{\lambda_i}^{-1}(W_{\lambda_i})$ is an ω −open set in X containing a and satisfying

$$cl(W) = \cap_{i=1}^{n} p_{\lambda_i}^{-1}(cl(W_{\lambda_i})) \subset \cap_{i=1}^{n} p_{\lambda_i}^{-1}(U_{\lambda_i}) \subset U.$$

Thus X is $\omega - T_3$.

Using the theorems, Theorem 2.17, Theorem 2.18, Theorem 2.19, Theorem 2.20, Theorem 2.21, Theorem 2.22 and the fact $cl_\omega(A \cap B) \subseteq cl_\omega(A) \cap cl_\omega(B)$ [3] we can prove the other cases ✜

3. WEAKLY ω −CONTINUITY AND REGULAR SPACES

Weak continuity due to Levine [7] is one of the most important weak forms of continuity in topological spaces. In this article we use the ω −open set to introduce a new class of functions called weakly ω −continuous and study its relationships with the class of continuous functions and the class of ω −continuous functions introduced by T. Noiri, A. Al-Omari, and M. S. M. Noorani in [4].

Definition 3.1: [4] Let (X,τ) and (Y,σ) be two topological spaces. A function $f: X \longrightarrow Y$, is said to be ω *−continuous* if $f^{-1}(V)$ is ω −open for each open set in Y.

Definition 3.2: Let (X,τ) and (Y,σ) be two topological spaces. A function $f: X \longrightarrow Y$, is said to be *weakly* ω *−continuous* if and only if for each x in X and each V in σ containing $f(x)$ there exists an ω −open set U containing x such that $f(U) \subset cl_\omega(V)$.

Remark 3.3: We can easily prove that any continuous function is ω −continuous, and the ω −continuous function is weakly ω −continuous.

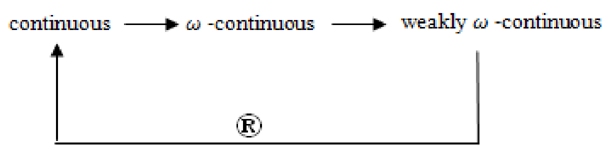
Example 3.4: Let $X = \{a,b,c\}, \tau = \{\phi, X, \{c\}\}$ and $\sigma = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$, let $f: (X,\tau) \longrightarrow (X,\sigma)$ be the identity function. Then f is weakly ω −continuous function but it is not continuous.

Remark 3.5: From above example we obtain that a weakly ω −continuous function needn't be continuous. However, we have:

Theorem 3.6: If $f: (X,\tau) \longrightarrow (Y,\sigma)$ is weakly ω −continuous function , X satisfies ω −condition and Y is ω_1 −regular, then f is continuous.

Proof: Let $x \in X$ and V any open set in Y containing $f(x)$. Then by Theorem 2.17 there is an open set W containing $f(x)$ and $f(x) \in W \subset cl_\omega(V) \subset V$. Since f is weakly ω −continuous function there is an ω −pen set U containing x such that $f(U) \subset cl_\omega(W) \subset V$. Then the ω −condition implies that □ is continuous ✜

By Remark 3.3 and Theorem 3.6 we obtain the following diagram:



® A type of regularity in Remark 3.7.

Fig. 2

Weakly ω −continuous diagram

Remark 3.7: By theorems: 2.16, 2.19 and 2.21 we can easily get that the above theorem still true for the ω −regular, ω_3 −regular and ω_5 −regular spaces. However the ω −condition on X makes Theorem 3.6 possible for each type of regularity here.

Theorem 3.8: Let $f: (X,\tau) \longrightarrow (Y,\sigma)$ be a weakly ω −continuous function and A a subset of X. if A is ω −open then $f|A: (A,\tau_A) \longrightarrow (Y,\sigma)$ is weakly ω −continuous.

Proof: It is clear that $\tau_\omega|A \subset (\tau|A)_\omega$. Since f is weakly ω −continuous for each x in A and each V in σ containing $f(x)$ there is U in τ_ω (the collection of all ω −open sets in τ) containing x such that $f(U) \subset cl_\omega(V)$. Then $x \in U \cap A \in (\tau|A)_\omega$ and $(f|A)(U \cap A) \subset cl_\omega(V)$. Thus $f|A$ is weakly ω −continuous ✜

Theorem 3.8: Let $f: (X,\tau) \longrightarrow (Y,\sigma)$ be a weakly ω −continuous function, and $g: (Y,\sigma) \longrightarrow (Z,\theta)$ be a continuous function, then $g \circ f: (X,\tau) \longrightarrow (Z,\theta)$ is weakly ω −continuous function.

Proof: Let V be an open set in Z, since g is continuous on Y so $g^{-1}(V)$ is open in Y. Then by the weakly ω −continuity of f there is an ω −open set U in X such that

$$f(U) \subset cl_\omega(g^{-1}(V)) \subset g^{-1}(cl_\omega(V)).$$

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This implies that

$$g(f(U)) \subset cl_{\omega}(V) \quad ✜$$

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