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Weighted Statistical Convergence

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ABSTRACT

In this paper, the concept of weighted statistical convergence which is defined and studied in [8] is modified and some inclusion relations of the set of modified weighted statistical convergent sequences with Nörlund-Cesaro summability and statistical convergence is given.

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1. INTRODUCTION

The concept of statistical convergence for real or complex valued sequence was introduced by H. Fast [1] and H. Steinhous [2] in the journal "Colloq. Math." independently in the same year 1951. Since then, this concept has been investigated by many authors such as [3], [4], [5], [6], [7] and etc.

Let K be a subset of $\mathbb{N}$ and $K(n):=\{k: k\leq n,\ k\in K\}$.

Then, the asymptotic density of K is denoted by $\delta(K)$, and defined as:

$$\delta(K):=\lim_{n\to\infty}\frac{1}{n}|K(n)|, \qquad (1.1)$$

If the limit exists. In (1.1), the vertical bars denote the cardinality of enclosed set.

A real valued sequence $x=(x_n)$ is said to be statistically convergent to the number L if for every $\varepsilon>0$, the set

$$K(n,\varepsilon):=\{k: k\leq n, |x_k - L|\geq\varepsilon\},$$

has asymptotic density zero, i.e.

$$\lim_{n\to\infty}\frac{1}{n}\{k: k\leq n, |x_k - L|\geq\varepsilon\}=0 \qquad (1.2)$$

and it is denoted by $x_n\to L(S)$.

It is clear from (1.2) that the idea of statistically convergence of the real (or complex) valued sequences completely depends on the asymptotic density of subset of natural numbers.

Also, some researchers generalized this notion by using some regular summability methods and they gave some analogy and new results [3], [4], [6], [8], [9], [10] and the others.

Let (p_n) be a positive sequence of real numbers such that

$$P_n := p_0 + p_1 + \ldots + p_n,$$

where $p_n\neq 0$ for all $n\in\mathbb{N}$ and $p_0>0$.

Then, the Nörlund transformation of (x_n) is defined as

$$t_n := \frac{1}{P_n}\sum_{k=0}^{n} p_k x_k .$$

If the sequence (t_n) has a finite limit L, then the sequence (x_n) is said to be $(\overline{N},p_n)-$ summable to L and it is denoted by $x_n\to L(\overline{N},p_n)$.

Also, if the limit of the sequence

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k\left|x_k - L\right|,$$

is zero when $n\to\infty$, then the sequence (x_n) is said to be strongly summable to L and it is denoted as $x_n\to L\left|\overline{N},p_n\right|$.

Let us note that if $P_n\to\infty$ when $n\to\infty$, then, Nörlund transformation is a regular summability method such that it transforms convergent sequences to convergent sequences and preserves the limit.

In [8], the concept of statistical convergence is generalized by using Nörlund summability method and it is called weighted statistical convergence.

Definition 1.1: [Definition 1, 8] A sequence $x=(x_n)$ is said to be weighted statistical convergence to L if for every $\varepsilon>0$,

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$$\lim_{n\to\infty}\frac{1}{P_n}\left|\{k : k \le n,\ p_k \mid x_k - L \mid \ge \varepsilon\}\right| = 0.$$

2. NEW RESULTS

2.1 Modification of Definition 1.1

The idea of statistical convergence of real or complex valued sequences completely depends on the asymptotic density of subset of natural numbers.

If $\delta(A)$ exists for any $A \subset \mathbb{N}$, then the equality $\delta(\mathbb{N} - A) = 1 - \delta(A)$ must be hold.
So, asymptotic density of natural numbers must be 1.

The main deficiencies of Definition 1.1 is weighted asymptotic density of natural numbers is not equal 1 in general.

If we consider the sequence (i) $(p_k) = (\frac{1}{k})$, we have:

$$\delta(\mathbb{N}) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^{-1}\left|\{k : k \le n, k \in \mathbb{N}\}\right| =$$

$$= \lim_{n\to\infty}\left(\ln n\right)^{-1}.n = \infty, \qquad (2.1)$$

(ii) $(p_k) = (k)$, we have

$$\delta(\mathbb{N}) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}k\right)^{-1}\left|\{k : k \le n, k \in \mathbb{N}\}\right| =$$

$$= \lim_{n\to\infty}\left(\frac{n(n+1)}{2}\right)^{-1}.n = 0 \qquad (2.2)$$

It is clear from (2.1) and (2.2) that we have to choose some a special (p_n) for $\delta(\mathbb{N}) = 1$.

From the discussion above, we should choose some (p_n) such that:

$$\lim_{n\to\infty}\frac{n}{P_n} = 1$$

is hold. Let us denote this by $P_n \approx n$.

For example, let us consider the sequence $(p_n) = (1 \mp \frac{1}{2^n})$ then $P_n = n \mp 1$ and $\delta(\mathbb{N}) = 1$.

Consequently, we can modify Definition 1.1 as follows:

Definition 2.1: (*Modified Weighted Statistical Convergence*) Let (p_n) be a positive sequence of real numbers such that $P_n \approx n$. A sequence $x = (x_n)$ is said to be weighted statistical convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{P_n}\left|\{k : k \le n,\ p_k \mid x_k - L \mid \ge \varepsilon\}\right| = 0.$$

It is denoted by $x_n \to L(S_{\bar{N}})$.

The set of weighted statistical convergent sequences is denoted by the symbol $S_{\bar{N}}$:

$$S_{\bar{N}} := \left\{x = (x_n) : x_n \to L(S_{\bar{N}}) \text{ for some } L\right\}.$$

It is clear from the **Definition 2.1** that the weighted statistical convergence and statistical convergence are coincide when $p_n = 1$.

2.2. New Results

In this section, some inclusion theorems should be given.

Theorem 2.1: Assume that $P_n \approx n$. Then, if $x_n \to L(\overline{N}, p_n)$ implies $x_n \to L(S_{\bar{N}})$.

Proof: Assume that $P_n \approx n$ and $x_n \to L(\overline{N}, p_n)$.

Denote the set $\left\{k : k \le n, p_k\left|x_k - L\right| \ge \varepsilon\right\}$ by $K(n,\varepsilon)$ for an arbitrary $\varepsilon > 0$. So, the inequality

$$\frac{1}{P_n}\sum_{k=1}^{n}\left|x_k - L\right| = \frac{1}{P_n}\left(\sum_{k\in K(n,\varepsilon)} + \sum_{k\notin K(n,\varepsilon)}\right)\left|x_k - L\right| \ge$$

$$\ge \frac{1}{P_n}\sum_{k\in K(n,\varepsilon)}\left|x_k - L\right| \ge \frac{1}{P_n}\sum_{k\in K(n,\varepsilon)}\varepsilon =$$

$$= \varepsilon\frac{1}{P_n}\left|\left\{k : k \le n, \left|x_k - L\right| \ge \varepsilon\right\}\right|$$

is hold. From the limit relation desired result is obtained.

Remark 2.1: The inverse of Theorem 2.1 is not true.

To see this, let us take $(p_n) = (1)$ and $x = (x_n)$ where

$$x_n = \begin{cases} m^3, & n = m^2, m \in \mathbb{N}, \\ 0, & n \ne m^2. \end{cases}$$

It is clear that $P_n \approx n$ and

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k : k \le n, 1.\left|x_k - 0\right| \ge \varepsilon\right\}\right| =$$

$$= \lim_{n\to\infty}\frac{1}{n}\left|\left\{k : k = m^2 \le n, 1.\left|m^3 - 0\right| \ge \varepsilon\right\}\right| = 0$$

is hold. But,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}1|x_k-0|=\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{[\|\sqrt{n}\|]}m^3=$$

$$=\lim_{n\to\infty}\frac{[\|\sqrt{n}\|]\left([\|\sqrt{n}\|]+1\right)\left(2[\|\sqrt{n}\|]+1\right)}{6n}=\infty.$$

Theorem 2.2: Assume that $P_n\approx n$ and there is a positive M such that $p_k|x_k-L|\le M$ for all $n\in\mathbb{N}$. Then, if $x_n\to L(S_{\overline{N}})$ implies $x_n\to L(\overline{N},p_n)$.

Proof: Denote by $K(n,\varepsilon)$ the set $\{k:k\le n,p_k|x_k-L|\ge\varepsilon\}$ for an arbitrary $\varepsilon>0$. From this notation we get,

$$\frac{1}{P_n}\sum_{k=1}^{n}p_k|x_k-L|=\frac{1}{P_n}\left(\sum_{k\in K(n,\varepsilon)}+\sum_{k\notin K(n,\varepsilon)}\right)p_k|x_k-L|\le$$

$$\le\frac{M}{P_n}\left|\{k:k\le n,p_k|x_k-L|\ge\varepsilon\}\right|+$$

$$+\frac{1}{P_n}\left|\{k:k\le n,p_k|x_k-L|<\varepsilon\}\right|.\varepsilon$$

So, we have $x_n\to L(\overline{N},p_n)$ since $x_n\to L(S_{\overline{N}})$.

Corollary 2.3: Under the conditions of Theorem 3.1, $x_n\to L(S_{\overline{N}})$ implies $x_n\to L(C,1)$ for $p_n\ge1$.

Proof: From Theorem 3.1 we have $x_n\to L(\overline{N},p_n)$. From the assumption, there exist positive constants m_1 and m_2 such that $m_1n\le P_n\le m_2n$. So, we get

$$\frac{1}{n}\sum_{k=1}^{n}(x_k-L)\le\frac{1}{n}\sum_{k=1}^{n}|x_k-L|\le\frac{m_2}{P_n}\sum_{k=1}^{n}p_k|x_k-L|.$$

After taking limit when $n\to\infty$, we obtained desired result.

Remark 2.2: The assumption $P_n\approx n$ is not omitted from the Theorem 2.2 and Corollary 2.3.

Take into consider $(p_k)=\left(\frac{1}{k}\right)$ and $(x_k)=\left(\sqrt{k}\right)$. It is clear that $P_n=\ln n$ and $P_n\approx n$ is not hold. In this case

$$\lim_{n\to\infty}\frac{1}{P_n}\left|\{k:k\le n,p_k|x_k-L|\ge\varepsilon\}\right|=$$

$$=\lim_{n\to\infty}\frac{1}{\ln n}\left|\left\{k:k\le n,\frac{1}{k}\left|\sqrt{k}-0\right|\ge\varepsilon\right\}\right|=$$

$$=\lim_{n\to\infty}\frac{1}{\ln n}\left|\left\{k:k\le n,\frac{1}{\sqrt{k}}\ge\varepsilon\right\}\right|=\lim_{n\to\infty}\frac{n_1(\varepsilon)}{\ln n}=0$$

when $n\to\infty$.

On the other hand we have

$$\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=0}^{n}p_k|x_k-L|=\lim_{n\to\infty}\frac{1}{\ln n}\sum_{k=0}^{n}\frac{1}{k}\left|\sqrt{k}-0\right|=$$

$$=\lim_{n\to\infty}\frac{1}{\ln n}\sum_{k=0}^{n}\frac{1}{\sqrt{k}}\ne0,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\sqrt{k}=\lim_{n\to\infty}\left(\sqrt{n}\right)\left(\sum_{k=0}^{n}\sqrt{\frac{k}{n}}\frac{1}{n}\right)=\lim_{n\to\infty}\frac{2\sqrt{n}}{3}\ne0$$

From the above discussion we have $\sqrt{k}\to0(S_{\overline{N}})$ but $\sqrt{k}\not\to0\left|\overline{N},p_n\right|$ and $\sqrt{k}\not\to0(C,1)$.

Theorem 2.4: Assume that $P_n\approx n$, for all $n\in\mathbb{N}$. Then the following statements are true:

(i). if $p_n\ge1$, for all $n\in\mathbb{N}$ and $x_n\to L(S_{\overline{N}})$ then, $x_n\to L(S)$.

(ii). if $p_n<1$, for all $n\in\mathbb{N}$ and $x_n\to L(S)$ then, $x_n\to L(S_{\overline{N}})$.

Proof: (i) if $p_n\ge1$, for all $n\in\mathbb{N}$ then, we have

$$\frac{1}{n}\left|\{k:k\le n,|x_k-L|\ge\varepsilon\}\right|\le$$

$$\le\frac{1}{n}\left|\{k:k\le n,p_k|x_k-L|\ge\varepsilon\}\right|\le$$

$$\le m_2\frac{1}{P_n}\left|\{k:k\le n,p_k|x_k-L|\ge\varepsilon\}\right|$$

For an arbitrary $\varepsilon>0$. So, we get

$$\frac{1}{n}\left|\{k:k\le n,|x_k-L|\ge\varepsilon\}\right|\to0,$$

when $n\to\infty$.

(ii) if $p_n<1$, for all $n\in\mathbb{N}$ then, we have

$$\frac{1}{P_n}\left|\{k:k\le n,p_k|x_k-L|\ge\varepsilon\}\right|\le$$

$$\leq \frac{1}{P_n}\left|\left\{k : k \leq n, \left|x_k - L\right| \geq \varepsilon\right\}\right| \leq$$

$$\leq \frac{1}{m_1}\frac{1}{n}\left|\left\{k : k \leq n, \left|x_k - L\right| \geq \varepsilon\right\}\right|$$

For an arbitrary $\varepsilon > 0$. So, we get

$$\frac{1}{P_n}\left|\left\{k : k \leq n, p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \to 0,$$

when $n \to \infty$.

Theorem 2.5: Assume that $P_n \approx n$, for all $n \in \mathbb{N}$. Then,

$$p_k\left|x_k - L\right| \to 0(S),\ n \to \infty,$$

If and only if $x_n \to L(S_{\overline{N}})$.

Proof: Take into consider the inequality

$$m_2 \frac{1}{P_n}\left|\left\{k : k \leq n, p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq$$

$$\geq \frac{1}{n}\left|\left\{k : k \leq n, p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq$$

$$\geq m_1.\frac{1}{P_n}\left|\left\{k : k \leq n, p_k\left|x_k - L\right| \geq \varepsilon\right\}\right|.$$

After taking limit we get desired result.

Corollary 2.6: Under the assumption of Theorem 2.5, If $p_k\left|x_k - L\right| \to 0,\ n \to \infty$, then $x_k \to L(S_{\overline{N}})$.

Proof: Since, the usual convergence implies statistical convergence and Theorem 2.4, the proof is obtained.

Remark 2.3: The inverse of Corollary 2.6 is not true.

For this, consider the sequence $x = (x_n)$ where

$$x_n = \begin{cases} k, & n = k^2, k \in \mathbb{N}, \\ \frac{1}{k}, & n \neq k^2. \end{cases}$$

and $(p_n) = (1 + 2^{-n})$. The assumption of Corollary 2.5 is hold. It is clear that $x = (x_n)$ convergent to zero in weighted statistical case but not in usual case.

Theorem 2.7: Assume that $P_n \approx n$, for all $n \in \mathbb{N}$ and $x = (x_n)$ is weighted statistical convergent to L. If $x_n = y_n$ a.a.k (in the usual sense), then the sequence (y_n) is weighted statistical convergent to L.

Proof: $x = (x_n)$ is weighted statistical convergent to L and $x_n = y_n$ a.a.k, that is the set $A = \{n : x_n \neq y_n\}$ has asymptotic density zero in the sense (1.1), i.e.,

$$\delta(A) = \lim_{n\to\infty}\frac{1}{n}\left|\left\{k : k \leq n, k \in A\right\}\right| = 0.$$

So we have,

$$\left\{k : k \leq n, p_k\left|y_k - L\right|\right\} =$$

$$= \left\{k : k \leq n, k \notin A, p_k\left|x_k - L\right|\right\} \cup$$

$$\cup \left\{k : k \leq n, k \in A,\ p_k\left|y_k - L\right|\right\}$$

And

$$\frac{1}{P_n}\left|\left\{k : k \leq n, p_k\left|y_k - L\right|\right\}\right| =$$

$$= \frac{1}{P_n}\left|\left\{k : k \leq n, k \notin A, p_k\left|x_k - L\right|\right\}\right| +$$

$$+ \frac{1}{P_n}\left|\left\{k : k \leq n, k \in A,\ p_k\left|y_k - L\right|\right\}\right|$$

$$\leq \frac{1}{m_1 n}\left|\left\{k : k \leq n, k \notin A, p_k\left|x_k - L\right|\right\}\right| +$$

$$+ \frac{1}{m_1 n}\left|\left\{k : k \leq n, k \in A,\ p_k\left|y_k - L\right|\right\}\right|$$

After taking limit when $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{1}{P_n}\left|\left\{k : k \leq n, p_k\left|y_k - L\right|\right\}\right| = 0.$$

This gives the desired result.

Theorem 2.8: Assume that $P_n \approx n$, for all $n \in \mathbb{N}$. A sequence $x = (x_n)$ is weighted statistical convergent to L if and only if there exists a monotone increasing sequence (k_n) such that $\delta\left(\left\{k_n : n \in \mathbb{N}\right\}\right) = 1$ and $\lim_{n\to\infty} p_{k_n}\left|x_{k_n} - L\right| = 0$.

Proof: "$\Leftarrow$" Suppose that there exists a monotone sequence (k_n) such that $\delta\left(\left\{k_n : n \in \mathbb{N}\right\}\right) = 1$ and $\lim_{n\to\infty} p_{k_n}\left|x_{k_n} - L\right| = 0$.

Then, for an arbitrary $\varepsilon > 0$, there is a positive $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $p_{k_n}\left|x_{k_n} - L\right| < \varepsilon$ for all

$n > n_0(\varepsilon)$. From this fact and Theorem 3.4 (i) the subsequence (x_{k_n}) is weighted statistical convergent to L.

$$\lim_{n\to\infty} p_{k_n}\left|x_{k_n} - L\right| = 0.$$

If we denote $K(\varepsilon) := \left\{k : p_k\left|x_k - L\right| \geq \varepsilon\right\}$ and $K^*(\varepsilon) = \left\{k_{n_0+1}, k_{n_0+2}, \ldots\right\}$ then it is clear that $\delta(K^*(\varepsilon)) = 1$ and $K(\varepsilon) \subset \mathbb{N} - K^*(\varepsilon)$. From this we get $\delta(K(\varepsilon)) = \delta(\mathbb{N}) - \delta(K^*(\varepsilon))$ and $x = (x_n)$ is weighted statistical convergent to L.

"$\Rightarrow$" Assume that the sequence $x = (x_n)$ is weighted statistical convergent to L.

Denote the set $K(\varepsilon) := \left\{k : p_k\left|x_k - L\right| \geq \varepsilon\right\}$ and $K^c(\varepsilon) := \left\{k : p_k\left|x_k - L\right| < \varepsilon\right\}$. So, from the assumption we have $\delta(K(\varepsilon)) = 0$ and $\delta(K^c(\varepsilon)) = 1$ for all $\varepsilon > 0$. If we take $\varepsilon = \frac{1}{r}$ we have also $\delta(K(\frac{1}{r})) = 0$ and $\delta(K^c(\frac{1}{r})) = 1$ for all $r \in \mathbb{N}$.

It is clear that:

$$K^c(1) \supset K^c(\frac{1}{2}) \supset \ldots \supset K^c(\frac{1}{r}) \supset K^c(\frac{1}{r+1}) \supset \ldots$$

Let us begin to choose k_1 which is the minimal element of $K^c(1)$, and we can choose $k_2 \geq k_1$ which is the minimal element of $K^c(\frac{1}{2})$.

Otherwise the number of elements of $K^c(\frac{1}{2})$ is at most k_1.

This is a contradiction to the assumption on $K^c(\frac{1}{2})$. If continue this process we obtain monotone increasing sequence (k_n) such that:

4. CONCLUSION

In this paper, we improved some deficiencies from the original definition of weighted statistical convergence which was given in [8].
After then, some inclusion results have been successfully given by using new modified definition of weighted statistical convergence.

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