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Application of New Homotopy Analysis Method for First and Second Orders Integro- Differential Equations

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ABSTRACT

This paper is aimed to demonstrate the modified form of Homotopy Analysis Method for solving both linear and nonlinear integro differential equations.
The modified form of Homotopy Analysis method is found to be fast, reliable and accurate and contains an auxiliary parameter that provides a powerful tool to analyses strongly linear and nonlinear (without linearization) problems directly. Numerical examples are presented to compare the results obtained with some existing results found in literatures. Results obtained by the modified form Homotopy Analysis method are compared favorably and performed better in terms of accuracy achieved.

Keywords: *Homotopy, Perturbation, Sine-Cosine wavelets, integro-differential equation*

1. INTRODUCTION

The Homotopy Analysis Method (HAM) proposed by Liao [12-14] is a general analytic approach to solve various types of linear and nonlinear equations, including partial differential equations, ordinary differential equations, differential difference equations, algebraic equations. More importantly, different from all perturbation and traditional non-perturbation methods. The HAM provides a simple way to ensure the convergence of solution in series form and therefore, the HAM is valid even for strong nonlinear problems.

The Homotopy Analysis Method (HAM) is based on homotopy, a fundamental concept in topology and differential geometry. Briefly, in HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of the problems to be considered. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure convergence of solution series. The method enjoys great freedom in choosing initial approximation and auxiliary linear operator. By means of this kind of freedom, a complicated non-linear problems can be transformed into an infinite numbers of simpler linear sub-problems.

In this paper, we present an iterative scheme based on the New Homotopy Analysis Method (NHAM) for the following kind of integro-differential equation:

$$a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + \;........ + a_n = f(x) + \lambda\int_a^b k(x,t)\,y(t)\,dt \qquad (1)$$

Subject to the following conditions

$$y(a) = \alpha_1\,,\; y^{/}(a) = \alpha_2\,, \quad ..... \quad y^{(n-1)}(a) = \alpha_{n-1} \qquad (2)$$

Where a, α_i are real constants, n is a positive integer, $f(x)$ *and* $k(x,t)$ are given functions and y is an unknown function.

Eq. (1) and (2) occur in various areas of engineering, physics, chemistry etc. Many methods have been used to handle eq. (1) and (2). Among them are Adomian decomposition, CAS wavelets, Sine-Cosine wavelets (SCW), Homotopy perturbation and Direct method based on Fourier and Block pulse functions (see [1 – 7, 9]) to mention just a few.

In this paper, we developed a New Homotopy Analysis Method (NHAM) to solve this class of equations. We also show that the new Homotopy Analysis method is more efficient than the other existing methods.

2. ANALYSIS OF THE NEW HOMOTOPY ANALYSIS METHOD (NHAM) .

Consider the following nth order integro-differential equation of the form given in eq. (1) and (2) (see [8]).
For solving these eq. by NHAM, we construct the zeroth-order deformation as:

$$(1-q)L\left(\frac{\partial^n y(x,q,h}{\partial x^n} - f(x)\right) = hq\left[\frac{\partial^n y(x,q,h)}{\partial x^n} - f(x) - \lambda\int_a^b k(x,t)\,y(t,q,h)\,dt\right] \qquad (3)$$

For $q = 0$ *and* $q = 1$ eq.(3) becomes:

$$y(x,0,h) = L^{-n}\,(f(x)\,dx) \;=\; g(x) \qquad (4)$$

and

$$y(x,1,h) \;=\; y(x) \qquad (5)$$

The corresponding modified mth-order deformation eq. is given by

$$L[y_m(x) - X_m\, y_{m-1}(x)] = hH(x)\vec{R}_m(y_{m-1}(x)$$

(6)

Where,

$$R_m\left(\vec{y}_{m-1}(x)\right) = \frac{1}{(m-1)}\frac{\partial^{m-1}N[\phi(x,q)]}{\partial q^{m-1}}\Big|_{q=0}$$

(7)

And $X_m = \begin{cases} 0. & m \le 1 \\ 1, & m > 1 \end{cases}$

(8)

Using eq. (6)-(8), one can comfortably have

$$R_m\left(\vec{y}_{m-1}(x)\right) = y^{(n)}_{m-1}X_m - f(x)X_m - \lambda\int_0^1 k(x,t)\, y_{m-1}(t)\,dt \quad (9)$$

And

Applying L^{-1} on both sides of eq. (6) , we get

$$y_m(x) = (1 - X_m)y_{m-1}(x) + hL^{-1}\left[H(x)R_m(\vec{y}_{m-1}(x))\right]$$

(10)

In this way, it easy to obtain y_m *for* $m \ge 1$, *at mth – order, we have*

$$y(x) = y_0 + \sum_{m=1}^{\infty} y_m(x)$$

(11)

We then present a simple iterative scheme for $y_m(x)$. To this end, the linear operator L is chosen to be

$$L[y(x)] = \frac{d^n y}{d x^n}$$

as an initial guess $y_0 x) = g(x)$ is taken which is obtained by decomposing $g(x)$ in eq. (4) into two parts, namely:

$$g_0(x) \;\; and \;\; g_1(x)\;,\;\; i.e \;\; g(x) = g_0(x) + g_1(x)$$

(12)

Where $g_1(x)$ is assigned to the component $y_1(x)$ among other terms and auxiliary function $H(x) = 1$ are taken.

Thus, the following recursion formulae for the new Homotopy Analysis Method (NHAM) are formulated as:

$$y_0(x) = g_0(x)$$

$$y_1(x) = g_1(x) + hL^{-n}(R(\vec{y}_0(x))$$

:

:

$$y_m(x) = -hL^{-1}\left(R_m(\vec{y}_{m-1}(x)\right), \;\; m = 2, 3, ...$$

Where

$$L^{-1}(.) = \int_0^{\tau_0}\int_0^{\tau_1}..................\int_0^{\tau_{n-1}}(.)\,d\tau..........d\tau$$

Eq. (13) demonstrates reliability in that it accelerates the convergence of the solution and reduces the series of computation as compared to perturbation techniques.

3. NUMERICAL EXAMPLES

In order to investigate the accuracy of the NHAM solution with a finite number of terms, three examples were solved.

We define absolute error as:

$$|Ey_{NHAM}| = |y_{exact} - y^N_{NHAM}|$$

Note: Matlab 7. is used to carry out the computations.

Example 1.

Consider the following integro – differential equation:

$$y'(x) = 3e^{3x} - \frac{1}{3}(2e^3 + 1)x + \int_0^1 3xt\, y(t)dt, \qquad y(0) = 1$$

The exact solution is $y(x) = e^{3x}$

By the NHAM, we constructed the zeroth-order deformation (3). Thus, by using eq. (4), we obtain

$$g(x) = e^{3x} - \frac{1}{6}(2e^3 + 1)x^2$$

Let

$$g_0(x) = x - \frac{1}{6}(2e^3 + 1)x^2 \quad and \quad g_1(x) = e^{3x} - x$$

And using the recursive relations (13) , we find

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$$y_0(x) = x - \frac{1}{6}\left(2e^3 + 1\right)x^2$$

$$y_1(x) = e^{3x} - x - h\int_0^1 \left(\int_0^1 3\tau t\, y_0(t)dt\right)d\tau$$

:

:

$$y_m(x) = -h\int_0^1\left(\int_0^1 3\tau t\, y_{m-1}(t)\,dt\right), \quad m = 2,3$$

Some numerical results are given in table 1 below.

Table 1: Numerical Results of Example 1: (NHAM: h =-1)

X	$\lvert y^{10}_{NHAM}\rvert$	$\lvert y^{20}_{NHAM}\rvert$	SCW[10]	HPM[9]
0.03125	1.28131E-06	1.0000E-09	3.0000E-04	1.38192E-07
0.15625	3.19899E-05	1.0000E-09	1.0000E-03	3.45482E-06
0.28125	1.03651E-04	9.0000E-09	2.1000E-03	1.11936E-05
0.40625	2.16256E-04	1.4000E-08	3.0000E-03	2.33546E-05
0.53125	3.69809E-04	2.2000E-08	3.2000E-03	3.99377E-05
0.65625	5.64311E-04	3.4000E-08	4.7000E-03	6.09430E-05
0.78125	7.99750E-04	4.0000E-08	6.2000E-03	8.63704E-05
0.90625	1.07610E-04	6.0000E-08	8.4000E-03	1.16220E-04

Example 2:

Consider the following linear fredholm integro- differential equation

$$y'(x) = 1 - \frac{x}{3} + \int_0^1 xt\, y(t)dt, \qquad y(0) = 0$$

The exact solution is $y(x) = x$

By NHAM, we constructed the zeroth-order deformation (4) .

Thus, by using eq. (3), we obtain

$$g(x) = x - \frac{x^2}{6}$$

Let $g_0(x) = x - 1$ *and* $g_1(x) = 1 - \frac{x^2}{6}$

We got the components in the following form:

$$y_0 = x - 1$$

$$y_1(x)\ 1 - \frac{x^2}{6} - h\int_0^1\left(\int_0^1 \tau t y_0(t)\,dt\right)d\tau$$

:

:

$$y_m(x) = -h\int_0^1\left(\int_0^1 (\tau t y_{m-1}(x)dt\right)d\tau, \qquad m = 2,3$$

Some numerical results of these solutions are presented in table 2 below (NHAM:h=-1)

Table 2: Numerical Results of Example 2

X	$\lvert y^{5}_{NHAM}\rvert$	$\lvert y^{10}_{NHAM}\rvert$	Method [5] with degree 5	CAS Wavelet [7]
0.1	6.1100E-07	1.0000E-09	2.06509E-04	2.1794238E-04
0.2	6.4420E-06	1.0000E-09	8.04069E-04	6.3854821E-04
0.3	5.4940E-06	1.0000E-09	1.72624E-03	7.9137049E-04
0.4	9.7660E-06	1.0000E-09	2.86044E-03	2.1558601E-02
0.5	1.5259E-05	1.0000E-09	4.04527E-03	4.9935843E-03
0.6	2.1973E-05	1.0000E-09	5.06663E-03	2.2172881E-02
0.7	2.9908E-05	2.0000E-09	5.65279E-03	1.0564545E-04
0.8	3.9064E-05	3.0000E-09	5.46844E-03	1.4323368E-03
0.9	4.9439E-05	2.0000E-09	4.10753E-03	2.0774746E-02

Example 3:

We now consider the second order integro-differential equation with constant coefficients.

$$y''(x) = e^x - x + \int_0^1 xt\, y(t)\,dt, \qquad y(0) = 1, \quad y'(0) = 0$$

The exact solution is $y(x) = e^x$.

The problem is solved by the same applied in examples 1 and 2.

Therefore, the iterations formulation are follows:

$$y_0(x) = e^x - x$$

$$y_1(x) = x - \frac{x^3}{6} - h\int_0^1\int_0^1\left(\int_0^1 \tau\, t\,(e - t)dx\right)dt\, d\tau$$

:

:

$$y_m(x) = -h\int_0^1\int_0^1\left(\int_0^1 \tau\, t\, y_{m-1}(t)dt\right)d\tau d\tau, \qquad m = 2, 3$$

Some numerical results of these solutions are presented in table 3.

Table 3: Numerical Results of Examples 3 when h = -1

X	$\left\|y^{5}_{NHAM}\right\|$	$\left\|y^{10}_{NHAM}\right\|$
0.1	2.0000E-09	0.0000E+00
0.2	1.0000E-09	0.0000E+00
0.3	2.0000E-09	0.0000E+00
0.4	4.0000E-09	0.0000E+00
0.5	9.0000E-09	1.0000E-09
0.6	1.0000E-08	0.0000E+00
0.7	2.3000E-08	0.0000E+00
0.8	3.6000E-08	1.0000E-09
0.9	5.0000E-08	0.0000E+00
1.0	6.9000E-08	0.0000E+00

4. CONCLUSION

The new Homotopy analysis method has been successfully developed and new to solve first and second orders integro – differential equations. The examples analyzed illustrate the ability and reliability of the developed method and reveals that the method is very simple and effective. The new method compared favorably and better in terms of accuracy achieved. The results indicate that the convergence rate is very fast and lower approximations can achieve high accuracy.

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