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Asymptotic Solution of Partial Differential Equations Depending on a Small Parameter

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ABSTRACT

We construct asymptotic solution of a partial differential equation with small parameter. We have proven the solution is unique and uniform in the domain Ω and, further, the asymptotic approximation is within $O\left(\varepsilon^2\right)$.

Keywords: Asymptotic solution , Differential equation

1. INTRODUCTION

Asymptotic solutions of partial differential equations have been intensively investigated. Some results on the construction of asymptotics for solutions of singularly perturbed problems with first-order partial derivatives have been obtained [1-4]. Asymptotic expansions of periodic solutions of second –and third- order ordinary differential equations of arbitrary order were constructed in [5, 6]. In [7], the researcher studied a second order elliptic equation with a small parameter at one of the highest order derivatives is considered in a three-dimensional domain, the limiting equation is a collection of two-dimensional elliptic equations in two-dimensional domains depending on one parameter. In [8],Considered it was that a mixed boundary-initial value problems for a partial differential equation in the critical case for the singular perturbed, and constructed the asymptotic expansion for the modified problem and proved the uniqueness of the solution. In [9] constructed and justified the asymptotic solutions as two series in the powers of small parameters consisting four parts and derivatives degenerate into the systems of partial differential equation of first order, he proved that the solution is uniform in the domain. Also he proved a unique solution and the asymptotic approximation is within $O(\varepsilon^{n+1})$. In [10], studied systems of non-linear ordinary differential equations are considered in the semi-infinite interval, the coefficients of the equations can have infinite upper limits as $t\rightarrow\infty$, and the theorems of the existence and uniqueness of the solutions of such singular Cauchy problems with continuous dependence of these solutions on the singularly large parameter occurring in the equations is investigated. The initial value problem for a system of nonlinear ordinary differential equations with a small parameter multiplying the highest derivative is investigated, in a neighbourhood of the initial point the asymptotic behaviour of the solution has quite a complicated structure, In [11]. The asymptotic behaviour of solutions of the first boundary-value problem for a second-order elliptic equation in a domain with angular points is investigated for the case when a small parameter is involved in the equation only as a factor multiplying one of the highest order derivatives and the limit equation is an ordinary differential equation, In[12]. In [13], studied a modification of an initial-boundary–value problem in the critical case for the heat-conduction equation in a thin domain, and he justified asymptotic expansions of the solutions of the problems with respect to a small parameter $\varepsilon > 0$,he proven that the solution is uniform in the domain and the asymptotic approximation is within $O(\varepsilon^{n+1})$. In[14], obtained an asymptotic expansion, containing regular boundary corner functions in the small parameter ε , for the solution of a second order partial differential equation, he constructed the asymptotic expansion $u_n(x,t,\varepsilon)$ for the modified problem and proved that it is the unique solution. Also he have proved that the solution is valid uniformly in the domain Ω and the asymptotic approximation is within $0(\varepsilon^{n+1})$.

A singularly perturbed system of two second-order differential equations (one rapid and one slow), was considered in [15]. In[16], a completed asymptotic expansion is constructed for solutions of the Cauchy problem for n^{th} order linear ordinary differential equations with rapidly oscillating coefficients, some of which may be proportional to $\omega^{n/2}$,where ω is oscillation frequency, a similar problem is solved for a class of systems of n linear first-order ordinary deferential equations with coefficients of the same type. Attention is also given to some classes of first-order nonlinear equations with rapidly oscillating terms proportional to powers ω^d For such equations with $d \in (1/2, 1]$, conditions that allow for the construction (and strict justification) of the leading asymptotic term were found and, in some cases, asymptotic expansion of the solution of the Cauchy problem has been completed. In the present paper, we consider asymptotic solution of a partial differential equation with initial- boundary value conditions.

2. SETTING THE PROBLEM

We consider, smoothing procedure used in constructing the asymptotic solution equation depending on the small parameter $\varepsilon > 0$ in the domain $(x,t) \in \Omega = (0 \le x \le 1) \times (0 \le t \le T)$ of the following:

$$\varepsilon \frac{\partial u}{\partial t} - \varepsilon^2 a(x,t) \frac{\partial^2 u}{\partial x^2} + b(x,t) \frac{\partial u}{\partial x} = f_0(u,x,t,\varepsilon) + \varepsilon f_1(u,x,t,\varepsilon) \quad (2\text{-}1)$$

$$u|_{t=0} = \varphi(x), \quad u|_{x=0} = \omega_0(t), \quad \left.\frac{\partial u}{\partial x}\right|_{x=0,1} = 0. \quad (2\text{-}2)$$

The functions $a(x,t),\ b(x,t),\ \varphi(x), f_0(u,x,t,\varepsilon) \text{ and } f_1(u,x,t,\varepsilon)$ are continuous and infinitely differentiable with respect to each of their arguments. In what follows we construct and justify an asymptotic expansion of initial-boundary-value type for the solution of the problem (2-1) and (2-2), subject to the following requirements conditions:

I. *The functions* $a(x,t) > 0$ *and* $b(x,t) > 0$, *for all* $(x,t) \in \Omega = (0 \le x \le 1) \times (0 \le t \le T)$. *Consistency of initial and boundary conditions are not expected, in particular* $\varphi'(0) \ne 0$.

II. *Let equation* $f(\bar{u}_0(0,t), x, t, 0) = 0$ *has a solution (root), which exists,* $\bar{u}_0(0,t) = \alpha_0(t)$. *In which connection* $\frac{\partial}{\partial u} f(\alpha_0(t), 0, t, 0) < 0,\ 0 \le t \le T$.

III. *The auxiliary problem*

$$\frac{dk}{d\tau} = f(k,0,0,0),\ \tau \ge 0,\ k(0) = \varphi(0), \quad (2\text{-}3)$$

where $\varphi(0)$ *the initial value of function* $\varphi(x)$. *Point* $k = \alpha_0(0)$ *is due to condition I and is the asymptotically stable rest point of the equation (2-3).*

IV. *Let value* $\varphi(0)$ *belongs to the domain of influence of the rest point* $k = \alpha_0(0)$, *i.e., the solution* $k(\tau)$ *of problem (2-3) exist for* $\tau \ge 0$ *and* $k(\tau) \to \alpha_0(0)$ *with* $\tau \to \infty$.

3. ALGORITHM FOR CONSTRUCTION ASYMPTOTIC FORM

We formulate an asymptotic expansion of the solution of problem (2-1), (2-2) in the form

$$u(x,t,\varepsilon) = \sum_{k=0}^{1} \varepsilon^k [\bar{u}_k(x,t) + v_k(x,\tau) + p_k(\xi,t) + q_k(\xi,\tau)], \quad (3\text{-}1)$$

where $\tau = \frac{t}{\varepsilon},\ \xi = \frac{x}{\varepsilon}$; $\bar{u}_k$ is the regular part of the asymptotic form, $v_k(x,\tau)$ and $p(\xi,t,\varepsilon)$ are ordinary boundary functions, which play a role in the vicinity of the side $t = 0, x = 0$ and $x = 1$ and $t = 0$ respectively; $q(\xi,\tau,\varepsilon)$ is angular boundary function which play a role in a neighbourhood of the corner point $(0,0)$.

We substitute series (3-1) into equation (2-1) and equate the coefficients of the same powers of ε in the left and right-hand sides of the obtained relations for each of the eight classes of functions in (3-1). Thus, we obtain problem for coefficients in (3-1).

4. REGULAR COEFFICIENT OF THE ASYMPTOTIC

We have problem for the regular coefficient of the asymptotic expansion (3-1), It is mean that $\bar{u}(x,t,\varepsilon)$, which is determined with the help of the equation:

$$\varepsilon \frac{\partial \bar{u}}{\partial t} - \varepsilon^2 a(x,t) \frac{\partial^2 \bar{u}}{\partial x^2} + b(x,t) \frac{\partial \bar{u}}{\partial x} = f_0(\bar{u},x,t,\varepsilon) + \varepsilon f_1(\bar{u},x,t,\varepsilon) \quad (4\text{-}1)$$

4.1 Regular coefficient of $\bar{u}_0(x,t)$

From equation (4-1), we obtain the following problems for the regular coefficient $\bar{u}_0(x,t)$

$$b(x,t) \frac{\partial}{\partial x} \bar{u}_0(x,t) = f_{00}(\bar{u}_0, x, t, 0), \quad 0 \le x \le 1$$
$$\frac{\partial}{\partial x} \bar{u}_0(0,t) = 0. \quad (4\text{-}2)$$

We consider equation (4-2) and taking into account condition II, we obtain the equation (4-2) with initial condition $\bar{u}_0(0,t) = \alpha_0(t)$.

4.2 Regular coefficient of $\bar{u}_1(x,t)$

The regular coefficient of $\bar{u}_1(x,t)$, we have the following problems,

$$\frac{\partial \bar{u}_0}{\partial t}+b(x,t)\frac{\partial \bar{u}_1}{\partial x}=\frac{\partial}{\partial u}f_{01}(\bar{u}_0,x,t,0)\bar{u}_1+f_{00}\left(\bar{u}_0(x,t),x,t,0\right)+f_{10}(\bar{u}_0,x,t,0) \quad (4\text{-}3)$$

$$b(x,t)\frac{\partial \bar{u}_1}{\partial x}=\frac{\partial}{\partial u}f_{01}(\bar{u}_0,x,t,0)\bar{u}_1+\Delta_1(x,t) \quad (4\text{-}4)$$

$$\frac{\partial}{\partial x}\bar{u}_1(x,t)=0, \quad (4\text{-}5)$$

where

$\Delta_1(x,t)=f_{00}\left(\bar{u}_0(x,t),x,t,0\right)+f_{10}\left(\bar{u}_0(x,t),x,t,0\right)-\frac{\partial \bar{u}_0}{\partial t}$ known function. For equation (4-4), when x=0, we obtain: and by using the condition $\frac{\partial}{\partial x}\bar{u}_1(0,t)=0$, we find the initial value $\bar{u}_1(x,t)$:

$$b(0,t)\frac{\partial \bar{u}_1}{\partial x}=\frac{\partial}{\partial u}f_{01}(\bar{u}_0,0,t,0)\bar{u}_1+\Delta_1(0,t),$$

is linear differential equation

$$\bar{u}_1(0,t)=-\frac{\Delta_1(0,t)}{\frac{\partial}{\partial u}f_{01}\left(\bar{u}_0(0,t),0,t,0\right)}=-\frac{f_{00}\left(\bar{u}_0(0,t),0,t,0\right)+f_{10}\left(\bar{u}_0(0,t),0,t,0\right)-\frac{\partial \bar{u}_0}{\partial t}}{\frac{\partial}{\partial u}f_{01}\left(\bar{u}_0(0,t),0,t,0\right)}$$

And by condition II we have:

$$\bar{u}_1(0,t)=-\frac{f_{00}\left(\alpha_0(t),0,t,0\right)+f_{10}\left(\alpha_0(t),0,t,0\right)-\frac{\partial \alpha_0(t)}{\partial t}}{\frac{\partial}{\partial u}f_{01}\left(\alpha_0(t),0,t,0\right)}.$$

With this initial condition, the solution of equation (4-4) is uniquely determined.

5. BOUNDARY COEFFICIENT IN NEIGHBOURHOOD OF THE INITIAL TIME INSTANT

We construct the following group of coefficients of series (3-1), the boundary function $v_0(x,\tau)$ and $v_1(x,\tau)$. Consider problem (2-1), (2-2) in a neighbourhood of the upper boundary $(t=0)$ of the domain Ω and perform the change of variables $\tau=\varepsilon t$; we have the problem,

$$\frac{\partial v}{\partial \tau}-\varepsilon^2 a(x,\varepsilon\tau)\frac{\partial^2 v}{\partial x^2}+b(x,\varepsilon\tau)\frac{\partial v}{\partial x}=f_0(v,x,\varepsilon\tau,\varepsilon)+\varepsilon f_1(v,x,\varepsilon\tau,\varepsilon) \quad (5\text{-}1)$$

$$v\big|_{\tau=0}=\varphi(x), \quad \frac{\partial v}{\partial x}\Big|_{x=0,1}=0 \quad (5\text{-}2)$$

5.1 The boundary function of $v_0(x,\tau)$

For $v_0(x,\tau)$ we get the problem:

$$\frac{\partial v_0}{\partial \tau}+b(x,0)\frac{\partial v_0}{\partial x}=f_{00}\left(\bar{u}_0(x,0)+v_0(x,0),x,0,0\right)-f_{00}\left(\bar{u}_0(x,0),x,0,0\right),$$
$$(x,\tau)\in\Omega=(0\le x\le 1)\times(0\le\tau<\infty), \quad (5\text{-}3)$$

$$v_0(x,0)=\varphi(x)-\bar{u}_0(x,0), \quad (5\text{-}4)$$

$$\frac{\partial v_0(0,\tau)}{\partial x}=0\,. \quad (5\text{-}5)$$

Characteristic $\tau=s(x)=\int_0^x\frac{dr}{b(r,0)}$, emerging from the corner point $(0,0)$, divides the region Ω into two parts. At $\tau\le s(x)$ solution determined by the condition (5-4) and we believe that this solution exists, and denote it by $v_0(x,\tau)$. For the solution for $\tau\ge s(x)$, we find first $v_0(0,\tau)$. For this purpose, in equation (5-3) we put $x=0$ and use condition (5-5), we have equation for $v_0(0,\tau)$ that is

$$\frac{\partial v_0(0,\tau)}{\partial \tau}=f_{00}\left(\bar{u}_0(0,0)+v_0(0,\tau),0,0,0\right), \quad \tau\ge 0\,.$$

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From (5-4) we have initial condition $v_0(0,0)=\varphi(0)-\bar{u}_0(0,0)$.

Changing $k=v_0(0,0)+\bar{u}_0(0,0)=v_0(0,\tau)+\alpha_0(0)$ reduces the problem to $v_0(0,\tau)$ the problem (2-3) in condition III. Due to condition IV, the solution $v_0(0,\tau)$ exist at $\tau\geq 0$, with $v_0(0,\tau)\to 0$ at $\tau\to\infty$. We denote this solution by $s(\tau)$. Inequality in condition II provides an exponential estimate for $s(\tau)$: $|s(\tau)|\leq c\,e^{-\beta\tau}$; here c and β positive constants, independent of ε.

To find $v_0(x,\tau)$ at $\tau\geq s(\tau)$ we need to solve equation (5-3) with condition $v_0(0,\tau)=s(\tau)$. We denote $v_0(x,\tau)$ at $\tau\geq s(x)$ by $v_0(x,\tau)$. Because of the exponential estimates for $s(\tau)$, the $v_0(x,\tau)$ have the estimate

$$|v_0(x,\tau)|\leq ce^{-\beta\tau},\qquad 0\leq x\leq 1,\ \tau\geq s(x). \qquad (5\text{-}6)$$

5.2 The boundary function of $v_1(x,\tau)$

For $v_1(x,\tau)$ we obtain the following problems:

$$\frac{\partial v_1}{\partial\tau}+b(x,0)\frac{\partial v_1}{\partial x}=\frac{\partial}{\partial v}f_{01}(v_0,x,0,0)v_1+f_{10}(v_0,x,0,0)$$

$$\frac{\partial v_1}{\partial\tau}+b(x,0)\frac{\partial v_1}{\partial x}=\frac{\partial}{\partial u}f_{01}\left(\bar{u}_0(x,0)+v_0(x,0),x,0,0\right)v_1+\varpi(x,\tau)$$

(5-7)

$$(x,\tau)\in\Omega=(0\leq x\leq 1)\times(0\leq\tau<\infty),$$

where $\tau=0$ we have

$$v_1(x,0)=-\bar{u}_1(x,0),\quad \frac{\partial v_1(0,\tau)}{\partial x}=0, \qquad (5\text{-}8)$$

where

$$\varpi(x,\tau)=f_{00}(\bar{u}_0+v_0,x,0,0)-f_{00}(\bar{u}_0,x,0,0)+\frac{\partial}{\partial u}f_{01}(\bar{u}_0,x,0,0)+f_{10}(v_0,x,0,0)-b(x,0)\frac{\partial v_0}{\partial x}.$$

Similarly, in case $v_0(x,\tau)$, the equation (5-7) and the conditions (5-8) so the $v_1(x,\tau)$ have the estimate $|v_1(x,\tau)|\leq ce^{-\beta\tau}$, $0\leq x\leq 1,\ \tau\geq s(x)$.

6. ORDINARY BOUNDARY COEFFICIENT $p_k(\xi,t,\varepsilon)$

The coefficients $p_k(\xi,t,\varepsilon)$ simultaneous with the regular coefficient must satisfy the boundary condition for $x=0$ and be determined from the problem

$$\varepsilon\frac{\partial p}{\partial t}+a(\varepsilon\xi,t)\frac{\partial^2 p}{\partial\xi^2}+b(\varepsilon\xi,t)\frac{\partial p}{\partial\xi}=f_0(p,\varepsilon\xi,t,\varepsilon)+\varepsilon f_1(p,\varepsilon\xi,t,\varepsilon)$$

(6-1)

and the conditions, when $\xi=0$ we have

$$p(0,t,\varepsilon)=\omega_0(t)-\bar{u}(0,t,\varepsilon),$$
$$\frac{\partial p}{\partial\xi}(0,t,\varepsilon)=0,$$

6.1 Ordinary Boundary function of $p_0(\xi,t)$

From this we obtain for $p_0(\xi,t)$ the problem

$$\left.\begin{aligned}&a(0,t)\frac{\partial^2 p_0}{\partial\xi^2}+b(0,t)\frac{\partial p_0}{\partial\xi}=f_{00}\left(p_0(0,t),0,t,0\right),\\&p_0(0,t)=\omega_0(t)-\bar{u}_0(0,t),\\&\frac{\partial p_0}{\partial\xi}(0,t)=0.\end{aligned}\right]$$

(6-2)

Note that t appears as a parameter $(0<t<T)$ and $\xi\geq 0$. The roots the corresponding characteristic equation $\lambda_{1,2}=-\frac{b(0,t)}{2}\mp\left[\frac{b^2(0,t)}{4}-f(p_0,0,t,0)\right]^{\frac{1}{2}}$ satisfy, by virtue of I, the condition $\operatorname{Re}\lambda_{1,2}<0$ for $0\leq t\leq T$.

Depending on whether $\mathrm{H}(t)=\frac{b^2(0,t)}{4}-f(p_0,0,t,0)$ is greater than, equal to, or less than zero, the solution (6-2) is written differently; however, in each of these cases it can, without difficulty, be written out explicitly and has an exponential estimate.

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Let

$$\lambda(t)=\begin{cases}\dfrac{b(0,t)}{2}-\sqrt{\mathrm{H}(t)}, & \text{if } \mathrm{H}(t)>0,\\ \dfrac{b(0,t)}{2}, \quad \sqrt{\quad} & \text{if } \mathrm{H}(t)\le 0.\end{cases}$$

It is clear that when $\mathrm{H}(t)\neq 0$ we can find a constant δ, or, when $\mathrm{H}(t)=0$, a first degree polynomial $\delta(\xi)$ with positive coefficients, such that the following estimate holds:

$$p_0(\xi,t)=(\Delta(0,t)+c_1)\exp(-\lambda(t)\xi),\quad 0\le t\le T,\ \xi\ge 0.$$

$$|p_0(\xi,t)|\le\delta(\xi)\exp(-\lambda(t)\xi),\quad 0\le t\le T,\ \xi\ge 0. \tag{6-3}$$

6.2 Ordinary Boundary function of $p_1(\xi,t)$

For $p_1(\xi,t)$ we have the problems:

$$\frac{\partial p_0}{\partial t}+a(0,t)\frac{\partial^2 p_1}{\partial\xi^2}+b(0,t)\frac{\partial p_1}{\partial\xi}=\frac{\partial}{\partial p}f_{01}(p(0,t),0,t,0)p_1+f_{10}(p_0(0,t),0,t,0)$$

$$a(0,t)\frac{\partial^2 p_1}{\partial\xi^2}+b(0,t)\frac{\partial p_1}{\partial\xi}=\frac{\partial}{\partial p}f_{01}(p(0,t),0,t,0)p_1+f_{10}(p_0(0,t),0,t,0)-\frac{\partial p_0}{\partial t},$$

$$p_0(0,t)=\omega_0(t)-\bar{u}_0(0,t),$$

$$\frac{\partial p_0}{\partial\xi}(0,t)=0.$$

Solution of these problems can also be written out in explicit form, namely,

$$|p_1(\xi,t)|\le\delta_1(\xi,t)\exp(-\lambda(t)\xi), \tag{6-4}$$

where $\delta_1(\xi,t)$ and its derivatives of arbitrary order increase as $\xi\to\infty$ no faster than some power of ξ. Because of this, the $p_1(\xi,t)$ will have an estimate of the form (6-3). This estimate shows $p_1(\xi,t)$ is actually boundary function in the variable ξ.

7. THE ANGULAR BOUNDARY COEFFICIENTS $p_k(\xi,\tau,\varepsilon)$

In this section, we study the angular boundary function $q(\xi,\tau,\varepsilon)$ appearing in the asymptotics of the solution because the $v(x,\tau,\varepsilon)$ function introduces a residual into the boundary conditions at $x=0$ and $x=1$, while the $p(\xi,t,\varepsilon)$ function introduce a residual into the initial condition at $t=0$. These residuals decay exponentially by virtue of (5-6) and (6-4), and have significance, therefore, only in small neighbourhoods of the points $(0,0)$ and $(1,0)$.The angular boundary function $q(\xi,\tau,\varepsilon)$ serves to eliminate these residuals in a neighbourhood of the point $(0,0)$, while the $p(\xi,t,\varepsilon)$ serves the purpose in a neighbourhood of the point $(1,0)$. For $q(\xi,\tau,\varepsilon)$, we consider problem (2-1) in a neighbourhood of the point $(0,0)$ of the domain Ω and perform the change of variables $x=\varepsilon\xi$ and $t=\varepsilon\tau$. We obtain the problem

$$\frac{\partial q}{\partial\tau}-a(\varepsilon\xi,\varepsilon\tau)\frac{\partial^2 q}{\partial\xi^2}+b(\varepsilon\xi,\varepsilon\tau)\frac{\partial q}{\partial\xi}=f_0(q(\varepsilon\xi,\varepsilon\tau,\varepsilon),\varepsilon\xi,\varepsilon\tau,\varepsilon)+\varepsilon f_1(q(\varepsilon\xi,\varepsilon\tau,\varepsilon),\varepsilon\xi,\varepsilon\tau,\varepsilon) \tag{7-1}$$

and the additional conditions

$$q(\xi,0,\varepsilon)=-p(\xi,0,\varepsilon)\ \ when\ \ \tau=0,\qquad \frac{\partial q}{\partial\xi}(0,\tau,\varepsilon)=-\frac{\partial v}{\partial x}(0,\tau,\varepsilon)\ \ when$$

$$q(0,\tau,\varepsilon)=-v(0,\tau,\varepsilon) \tag{7-2}$$

7.1 The angular Boundary function of $p_0(\xi,\tau,\varepsilon)$

From this, we obtain for $q_0(\xi,\tau)$ the problem

$$\frac{\partial q_0}{\partial\tau}-a(0,0)\frac{\partial^2 q_0}{\partial\xi^2}+b(0,0)\frac{\partial q_0}{\partial\xi}=f_{00}(q_0(0,0),0,0,0)$$

$$\xi>0\ \ and\ \tau>0 \tag{7-3}$$

$$q_0(\xi,0)=-p_0(\xi,0)\ \ when\ \ \tau=0,\qquad \frac{\partial q_0}{\partial\xi}(0,\tau)=-\frac{\partial v_0}{\partial x}(0,\tau)\ \ when\ \ \xi=0,$$

$$q_0(0,\tau)=-v_0(0,\tau).$$

Since $p_0(\xi,0)=[\omega_0(0)-\bar{u}_0(0,0)]\exp(-\lambda\xi)\equiv 0$, and, in exactly the same way, $v_0(0,\tau)\equiv 0$, it follows that $q_0(\xi,\tau)\equiv 0$.

7.2 The angular Boundary function of $p_1(\xi,\tau,\varepsilon)$

From the $q_1(\xi,\tau)$, we obtain the problem:

$$\frac{\partial q_1}{\partial\tau}-a(0,0)\frac{\partial^2 q_1}{\partial\xi^2}+b(0,0)\frac{\partial q_1}{\partial\xi}=\frac{\partial}{\partial q}f_{01}(q_0(0,0),0,0,0)q_1+f_{10}(q_0(0,0),0,0,0) \tag{7-4}$$

$$\frac{\partial q_1}{\partial\tau}-a(0,0)\frac{\partial^2 q_1}{\partial\xi^2}+b(0,0)\frac{\partial q_1}{\partial\xi}=\Lambda_1(\xi,\tau),$$
$$q_1(\xi,0)=-p_1(\xi,0),\qquad \frac{\partial q_1}{\partial\xi}(0,\tau)=-\frac{\partial v_1}{\partial x}(0,\tau),$$
$$q_1(0,\tau)=-v_1(0,\tau). \tag{7-5}$$

Where $\Lambda_1(\xi,\tau)$ are known functions. For the function $q_1(\xi,\tau)$ the following type of exponential estimate holds:

$$|q_1(\xi,\tau)|\le\begin{cases}\delta(\xi)\exp(-\beta\xi), & \xi\ge\tau\\ \delta(\tau)\exp(-\sigma\tau), & \xi\le\tau,\end{cases}$$

where $\delta(\xi)$ and $\delta(\tau)$ are polynomials with positive coefficients.

8. ESTIMATE OF THE REMAINDER TERMS

We denote by U_2 the 2-nd partial sums of the series (3.1).

Theorem: *The solution $u(x,t,\varepsilon)$ of the problem (2-1) and (2-2) are valid for small ε uniformly in the domain $\Omega=(0\le x\le 1)\times(0\le t\le T)$:*

$$u(x,t,\varepsilon)-U_2(x,t,\varepsilon)=O(\varepsilon^2).$$

Proof: Let $w=u-V$. If we substitute $u=V+w$ in problem (2-1) and (2-2), we have, for all remainder terms w, the following problem

$$\varepsilon\frac{\partial w}{\partial t}+\varepsilon^2 a(x,t)\frac{\partial^2 w}{\partial x^2}+b(x,t)\frac{\partial w}{\partial x}-f_u(x,t,\varepsilon)=W_0(w,x,t,\varepsilon)+\varepsilon W_1(w,x,t,\varepsilon), \tag{8-1}$$

where $f_u(x,t,\varepsilon)=f_u(V(x,t,\varepsilon),x,t,\varepsilon)$ and

$$W_0(w,x,t,\varepsilon)=f_0(V+w,x,t,\varepsilon)-f_u(x,t,\varepsilon)-\varepsilon\frac{\partial V}{\partial t}+\varepsilon^2 a(x,t)\frac{\partial^2 V}{\partial x^2}-b(x,t)\frac{\partial V}{\partial x}$$
$$W_1(w,x,t,\varepsilon)=f_1(V+w,x,t,\varepsilon)$$

The function $W_0(w,x,t,\varepsilon)$ *and* $W_1(w,x,t,\varepsilon)$ has the following two properties:

1- $W_0(w,x,t,\varepsilon)=O(\varepsilon^2)$ *and* $W_1(w,x,t,\varepsilon)=O(\varepsilon^2)$ uniformly in Ω, that follows from the fact that $V(x,t,\varepsilon)$ satisfies the equation (2.1) with an accuracy $O(\varepsilon^2)$.

2- For all number $c_1>0$, there exist numbers $c_2>0$ and $\varepsilon_0>0$ such that, if $|w_1|\le c_1\varepsilon,\ |w_2|\le c_2\varepsilon,\ 0<\varepsilon\le\varepsilon_0$, then

$$\left.\begin{array}{l}\max\limits_{\Omega}|W_0(w_1,x,t,\varepsilon)-W_0(w_2,x,t,\varepsilon)|\le c_2\varepsilon|w_1-w_2|\\ \text{and}\\ \max\limits_{\Omega}|W_1(w_1,x,t,\varepsilon)-W_1(w_2,x,t,\varepsilon)|\le c_2\varepsilon|w_1-w_2|\end{array}\right] \tag{8-2}$$

To prove property 2, we need to use the formula :

$$W_0(w_1,x,t,\varepsilon)-W_0(w_2,x,t,\varepsilon)=W_{w_0}(w_1-w_2)\ \text{and}\ W_1(w_1,x,t,\varepsilon)-W_1(w_2,x,t,\varepsilon)=W_{w_1}(w_1-w_2)$$

(derivative W_{w_0} *and* W_{w_1} is taken at an intermediate point) and evaluate

$$W_{w_{0,1}}=f_u(V+w,x,t,\varepsilon)-f_u(V,x,t,\varepsilon)=O(|w|).$$

Initial and boundary conditions for w are obtained by substituting $u=V+w$ in conditions (2-2). We have

$$w|_{t=0}=\phi(x,\varepsilon),\ \left.\frac{\partial w}{\partial x}\right|_{x=0}=\chi_0(t,\varepsilon),\ \left.\frac{\partial w}{\partial x}\right|_{x=1}=\chi_1(t,\varepsilon) \tag{8-3}$$

To prove the theorem, we must prove that the problem (8-1),(8-3) has a unique solution $w(x,t,\varepsilon)$, with $\max_{(x,t)\in\Omega}|w(x,t,\varepsilon)| = O(\varepsilon^2)$.

We make the change of unknown function $w = s\exp(\varpi(x))$, with function $\varpi(x)$ chosen so that the inequalities

$$b(x,t)\varpi'(x) > f_u(x,t,\varepsilon), \quad \varpi'(1) > 0, \quad \varpi'(0) < 0. \tag{8-4}$$

By virtue of condition II, a choice $\varpi(x)$ is obviously possible. Then for s we have problem:

$$\mathrm{H}_\varepsilon s \equiv \varepsilon\frac{\partial s}{\partial t} - \varepsilon^2 a(x,t)\frac{\partial^2 s}{\partial x^2} + b(x,t)\frac{\partial s}{\partial x} = \mathrm{M}_0(s,x,t,\varepsilon) + \varepsilon \mathrm{M}_1(s,x,t,\varepsilon), \tag{8-5}$$

$$s|_{t=0} = O(\varepsilon^2), \quad \left.\frac{\partial s}{\partial x} + \varpi'(0)s\right|_{x=0} = O(\varepsilon^2),$$

$$\left.\frac{\partial s}{\partial x} + \varpi'(1)s\right|_{x=1} = O(\varepsilon^2), \tag{8-6}$$

Where

$$\mathrm{M}_0(s,x,t,\varepsilon) = W_0(s\exp[\varpi(x)],x,t,\varepsilon)\exp[-\varpi(x)]$$

and

$$\mathrm{M}_1(s,x,t,\varepsilon) = W_1(s\exp[\varpi(x)],x,t,\varepsilon)\exp[-\varpi(x)]$$

.

The function M_0 *and* M_1 has the same properties 1 and 2 as the function W_0 *and* W_1. Applicable to the problem (8-5), (8-6) is the approximation Method:

$$s_0 = 0, \quad \Pi_\varepsilon s_n = \mathrm{M}_0(s_n,x,t,\varepsilon) + \mathrm{M}_1(s_n,x,t,\varepsilon)$$

The additional conditions for s_n are the same as in (8-6). For the first approximation s_1, we obtain problem

$$\Pi_\varepsilon s_1 = \mathrm{M}_0(0,x,t,\varepsilon) + \mathrm{M}_1(0,x,t,\varepsilon) = O(\varepsilon^2),$$

$$s_1|_{t=0} = O(\varepsilon^2), \quad \left.\frac{\partial s_1}{\partial x} + \varpi'(0)s_1\right|_{x=0} = O(\varepsilon^2),$$

$$\left.\frac{\partial s_1}{\partial x} + \varpi'(1)s_1\right|_{x=1} = O(\varepsilon^2).$$

As $\varpi'(1) > 0$, $\varpi'(0) < 0$, then for s_1 by the maximum principle, we obtain the estimate $\max_\Omega |s_1| \le c\varepsilon^2$, $c-$constant. This implies the existence and uniqueness of solution of the (8-1), (8-3) and we have the estimate $\max_\Omega |w| = O(\varepsilon^2)$. The proof of Theorem is complete.

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