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Upgrading Runge-Kutta-Fehlberg Method (RKFM) for Second Order Ordinary Differential Equations

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ABSTRACT

We modified the Runge-Kutta-Fehlberg method to a new direct and efficient method for general Second order Differential equation of the form

$$y'' = f(x, y, y')\ \ y(x_0) = y_0, y'(x_0) = y'_0$$

The linear Transformation T which is continuously differentiable on a set of a 3-tuple ordered pairs $(x, y, y') \rightarrow \mathbb{R}$ is adopted, this enable us to generate only Six functional evaluation which are directly and continuously used for the iterations of the solutions y_{n+i} . The new Scheme derived can handled General, Special, Light stiff and non-linear ODEs by efficiently.

Numerical Experiments are used to justify our claims.

Keywords: *Six order RKF Method, General, Special, non-linear and light Stiff ODEs*

1. INTRODUCTION

Many real life problems in Science and Engineering are often modeled in the form of Ordinary Differential equation of Second order. Such problems may not have closed form then numerical algorithms are usually employed to solve them

Effort have been made by researchers and solve higher order initial valued problems, especially Second order ODEs by a number of different method. Some numerical methods can be found in reference [1],[2] and [3].

In most of these method mentioned above reduction of the problems into system of first order ODEs (Indirect methods) were employed. The disadvantages of the indirect methods includes, too many auxiliary equations to be solve simultaneously, very long and complicated Computer programs, waste of both human and Computer time. Also some researchers developed a direct methods such as [4],[5],[6] and [7] though they avoided the method of reduction to system of First order ODEs. Their methods are good but are not very simple and efficient. The degree of accuracy is still low with their implementations.

Hence there is need to develop an algorithm which is superior, Simple and direct to handle the class of problem $y'' = f(x, y, y')\ \ y(x_0) = y_0, y'(x_0) = y'_0$. Since the current trend in numerical solution of differential equations is towards accuracy, efficiency and simple algorithms for Computer.

2. METHODOLOGY

We consider the general Second order ODEs of the form:

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x, y), y(x_0) = y_0, y'(x_0) = y'_0 \qquad (2.01)$$

where a_2, a_1, a_0 can be scalar constants or function of x and y

we write (2.0.1) in the form

$$\frac{d^2y}{dx^2} = f(x, y, y')\ y(x_0) = y_0, y'(x_0) = y'_0 \qquad (2.02)$$

We use the following Tabeau, constructed by E Felhberg for first order ODEs as basis for our work.

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C	A					
0	0					
$\frac{1}{4}$	$\frac{1}{4}$					
$\frac{3}{8}$	$\frac{3}{32}$	$\frac{9}{32}$				
$\frac{12}{13}$	$\frac{1932}{2197}$	$-\frac{7200}{2197}$	$\frac{7296}{2197}$			
1	$\frac{439}{219}$	-8	$\frac{3680}{513}$	$-\frac{845}{4104}$		
$\frac{1}{2}$	$-\frac{8}{27}$	2	$\frac{-3544}{2565}$	$\frac{1859}{4104}$	$\frac{-11}{40}$	
b^T	$\frac{16}{135}$	0	$\frac{6656}{12825}$	$\frac{28561}{56430}$	$-\frac{9}{50}$	$\frac{2}{55}$

(2.0.3)

Now we define a linear operator T on Set of ordered 3-tuples

$$\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k_j, y'\right)$$

as follows

$T(x,y,y') = y'$, $T'(x,y,y') = y'' = f(x,y,y')$ and more generally,

$$T'\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k'_j\right) = f\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k'_j\right)$$

where c_i ,a_{ij} are from table of coefficients.(Table 1)

$$T'\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k'_j\right) = \left(y' + h\sum_{j=1}^{s-1} a_{ij}k'_j\right)$$

$$k_i = T\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k_j, y'\right)$$

$$k'_i = T'\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k_j, y'\right)$$

We show that the Operator is well-defined.

Let z be a variable such that

$$T\left(x + c_i h, z + h\sum_{j=1}^{s-1} a_{ij}k_j, z' + h\sum_{j=1}^{s-1} a_{ij}k_j\right) = T\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k_j\right)$$

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and

$$T'\left(x + c_i h, z + h\sum_{j=1}^{s-1} a_{ij}k_j, z' + h\sum_{j=1}^{s-1} a_{ij}k_j, y'\right) = T'\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k_j, y'\right)$$

Then by the definition of T, we have

$$z' + h\sum_{j=1}^{s-1} a_{ij}k'_j = y' + h\sum_{j=1}^{s-1} a_{ij}k'_j \qquad (2.0.4)$$

and

$$f\left(x + c_i h, z + h\sum_{j=1}^{s-1} a_{ij}k_j, z' + h\sum_{j=1}^{s-1} a_{ij}k_j\right) = f\left(x + c_i h, y + h\sum_{j=1}^{s-1} a_{ij}k_j, y' + h\sum_{j=1}^{s-1} a_{ij}k_j\right) \qquad (2.0.5)$$

Combining (2.0.3) and (2.0.4) yields

$z' = y'$ and $y = z$

This shows that the operator T is unique and well-defined, we now generates the $k_i's$ as follows:

$$k_1 = T(x, y, y') = y' = L_1 \ , \ T'(x, y, y') = y'' = f(x, y, y') = m_1$$
$$k_2 = T\left((x + \tfrac{1}{4}h, y + \tfrac{1}{4}hk_1, y' + \tfrac{1}{4}hk'_1\right) = \left(y' + \tfrac{1}{4}hk'_1\right) = \left(y' + \tfrac{1}{4}hm_1\right) = L_1$$
$$k'_2 = T'\left(x + \frac{1}{4}h, y + \frac{1}{4}hk_1, y' + \frac{1}{4}hk'_1\right) = f\left(y' + \frac{1}{4}hk_1, y + \frac{1}{4}hk_1, y' + \frac{1}{4}hk'_1\right)$$
$$= f\left(x + \tfrac{1}{4}h, y + \tfrac{1}{4}hy', y' + \tfrac{1}{4}hm_1\right) = M_2$$
$$k_3 = T\left(x + \tfrac{3}{8}h, y + \tfrac{3}{32}hk_1 + \tfrac{9}{32}hk_2, y' + \tfrac{3}{8}hk'_1 + \tfrac{9}{32}hk'_2\right) = \left(y' + \tfrac{3}{8}hm_1 + \tfrac{9}{32}hm_2\right) = L_3$$
$$k'_3 = T'\left(x + \frac{3}{8}h, y + \frac{3}{32}hk_1 + \frac{9}{32}hk_2, y' + \frac{3}{8}hk'_1 + \frac{9}{32}hk'_2\right)$$
$$= f\left(x + \frac{3}{8}h, y + \frac{3}{8}hy' + \frac{9}{128}h^2m_1, y' + \frac{3}{32}hm_{1_1} + \frac{9}{32}hm_2\right) = m_3$$
$$k_4 = T\left(x + \frac{12}{13}h, y + \frac{1932}{2197}hk_1 - \frac{7200}{2197}hk_2 + \frac{7296}{2197}hk_3, y' + \frac{1932}{2197}hk'_1 - \frac{7200}{2197}hk'_2 + \frac{7296}{2197}hk'_3\right)$$
$$= \left(y' + \frac{1932}{2197}hm_1 - \frac{7200}{2197}hm_2 + \frac{7296}{2197}hm_3\right) = L_4$$
$$k'_4 = T'\left(x + \frac{12}{13}h, y + \frac{1932}{2197}hk_1 - \frac{7200}{2197}hk_2 + \frac{7296}{2197}hk_3, y' + \frac{1932}{2197}hk'_1 - \frac{7200}{2197}hk'_2 + \frac{7296}{2197}hk'_3\right)$$
$$= f\left(x + \frac{12}{13}h, y + \frac{1932}{2197}hy' - \frac{7200}{2197}h\left(y' + \frac{1}{4}hm_1\right) + \frac{7296}{2197}h\left(y' + \frac{3}{32}hm_1 + \frac{9}{32}hm_2\right), y' + \frac{1932}{2197}hm_1 - \frac{7200}{2197}hm_2 + \frac{7296}{2197}hm_3\right)$$
$$= f\left(x + \tfrac{12}{13}h, y + \tfrac{2028}{2197}hy' - \tfrac{1116}{2197}h^2m_1 + \tfrac{2052}{2197}h^2m_2, y' + \tfrac{1932}{2197}hm_1 - \tfrac{7200}{2197}hm_2 + \tfrac{7296}{2197}hm_3\right) = m_4$$
$$k_5 = T\left(x + h, y + \frac{439}{216}hk_1 - 8hk_2 + \frac{3680}{513}hk_3 - \frac{845}{4104}hk_4, y' + \frac{439}{216}hk'_1 - 8hk'_2 + \frac{3680}{513}hk'_3 - \frac{845}{4104}hk'_4\right)$$
$$= \left(y' + \frac{439}{216}hm_1 - 8hm_2 + \frac{3680}{513}hm_3 - \frac{845}{4104}hm_4\right) = L_5$$

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$$k'_5 = T'\left(x+h, y+\frac{439}{216}hk_1 - 8hk_2 + \frac{3680}{513}hk_3 - \frac{845}{4104}hk_4, y' + \frac{439}{216}hk'_1 - 8hk'_2 + \frac{3680}{513}hk'_3 - \frac{845}{4104}hk'_4\right)$$
$$= f\left(x+h, y+\frac{439}{216}hy' - 8h\left(y'+\frac{1}{4}hm_1\right) + \frac{3680}{513}h\left(y'+\frac{3}{32}hm_1+\frac{9}{32}hm_2\right) - \frac{845}{4104}h\left(y'+\frac{1932}{2197}hm_1 - \frac{7200}{2197}hm_2 + \frac{7296}{2197}hm_3\right), y' + \frac{439}{216}hm_1 - 8hm_2 + \frac{3680}{513}hm_3 - \frac{845}{4104}hm_4\right)$$
$$= f\left(x+h, y+hy' - \frac{353}{243}h^2m_1 + \frac{35}{13}h^2m_2 - \frac{80}{117}h^2m_3, y' + \frac{439}{216}hm_1 - 8hm_2 + \frac{3680}{513}hm_3 - \frac{845}{4104}hm_4\right) = m_5$$
$$k_6 = T\left(x+\frac{1}{2}h, y-\frac{8}{27}hk_1 + 2hk_2 - \frac{3544}{2565}hk_3 + \frac{1859}{4104}hk_4 - \frac{11}{40}hk_5, y' - \frac{8}{27}hk'_1 + 2hk'_2 - \frac{3544}{2565}hk'_3 + \frac{1859}{4104}hk'_4 - \frac{11}{40}hk'_5\right)$$

$$= \left(y' - \frac{8}{27}hm_1 + 2hm_2 - \frac{3544}{2565}hm_3 + \frac{1859}{4104}hm_4 - \frac{11}{40}hm_5\right)$$
$$k'_6 = T'\left(x+\frac{1}{2}h, y-\frac{8}{27}hk_1 + 2hk_2 - \frac{3544}{2565}hk_3 + \frac{1859}{4104}hk_4 - \frac{11}{40}hk_5, y' - \frac{8}{27}hk'_1 + 2hk'_2 - \frac{3544}{2565}hk'_3 + \frac{1859}{4104}hk'_4 - \frac{11}{40}hk'_5\right)$$
$$= f\left(x+\frac{1}{2}h, y-\frac{8}{27}hy' + 2h\left(y'+\frac{1}{4}hm_1\right) - \frac{3544}{2565}h\left(y'+\frac{3}{32}hm_1+\frac{9}{32}hm_2\right) + \frac{1859}{4104}h\left(y'+\frac{1932}{2197}hm_1 - \frac{7200}{2197}hm_2 + \frac{7296}{2197}hm_3\right) - \frac{11}{40}h\left(y'+\frac{439}{216}hm_1 - 8hm_2 + \frac{3680}{513}hm_3 - \frac{845}{4104}hm_4\right), y' - \frac{8}{27}hm_1 + 2hm_2 - \frac{3544}{2565}hm_3 + \frac{1859}{4104}hm_4 - \frac{11}{40}hm_5\right)$$

$$= f\left(x+\frac{1}{2}h, y+\frac{1}{2}hy' + \frac{4715}{22464}h^2m_1 + \frac{17}{52}h^2m_2 - \frac{3124}{6669}h^2m_3 - \frac{1859}{32832}h^2m_4, y' - \frac{8}{27}hm_1 + 2hm_2 - \frac{3544}{2565}hm_3 + \frac{1859}{4104}hm_4 - \frac{11}{40}hm_5\right) = m_6$$

The general solution of (2.0.1) is

$$y(x+h) = y + \frac{16}{135}hk_1 + 0hk_2 + \frac{6656}{12825}hk_3 + \frac{2856}{56430}hk_4 - \frac{9}{50}hk_5 + \frac{2}{55}hk_6$$

$$= y + \frac{16}{135}hl_1 + 0hl_2 + \frac{6656}{12825}hl_3 + \frac{2856}{56430}hl_4 - \frac{9}{50}hl_5 + \frac{2}{55}hl_6$$
$$= y + \frac{16}{135}hy' + \frac{6656}{12825}h\left(y'+\frac{3}{32}hm_1+\frac{9}{32}hm_2\right) + \frac{2856}{56430}h\left(y'+\frac{1932}{2197}hm_1 - \frac{7200}{2197}hm_2 + \frac{7296}{2197}hm_3\right) - \frac{9}{50}h\left(y'+\frac{439}{216}hm_1 - 8hm_2 + \frac{3680}{573}hm_3 - \frac{845}{4104}hm_4\right) + \frac{2}{55}h\left(y' - \frac{8}{27}hm_1 + 2hm_2 - \frac{3544}{2565}hm_3 + \frac{1859}{4104}hm_4 - \frac{11}{40}hm_5\right)$$
$$y(x+h) = y + hy' + \frac{2783}{23760}h^2m_1 + \frac{4352}{12825}h^2m_3 + \frac{2197}{41040}h^2m_4 - \frac{1}{100}h^2m_5 \qquad (2.0.6)$$

Substituting $x = x_n,\ y = y_n$ in (2.0.6) to obtain

$$y_{n+1} = y_n + hy'_n + \frac{2783}{23760}h^2m_1 + \frac{4352}{12825}h^2m_3 + \frac{2197}{41040}h^2m_4 - \frac{1}{100}h^2m_5 \qquad (2.0.7)$$
$$y'(x+h) = y' + \frac{16}{135}hm_1 + 0hm_2 + \frac{6656}{12825}hm_3 + \frac{2856}{56430}hm_4 - \frac{9}{50}hm_5 + \frac{2}{55}hm_6$$
$$y'_{n+1} = y'_n + \frac{16}{135}hm_1 + 0hm_2 + \frac{6656}{12825}hm_3 + \frac{2856}{56430}hm_4 - \frac{9}{50}hm_5 + \frac{2}{55}hm_6 \qquad (2.0.8)$$

where $m_1, m_2, m_3, m_4, m_5, m_6$ are already defined above.

Thus a Fifth order RK typed method for direct solution of (2.0.2) is

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$$y_{n+1} = y_n + hy'_n + \frac{1}{5}h^2\left(\frac{253}{432}m_1 + \frac{4352}{2565}m_3 + \frac{2197}{8208}m_4 - \frac{1}{20}m_5\right)$$
$$y'_{n+1} = y'_n + \frac{1}{5}h\left(\frac{16}{27}m_1 + \frac{6656}{2565}m_3 + \frac{2856}{11286}m_4 - \frac{9}{10}m_5 + \frac{2}{11}m_6\right) \quad (2.0.9)$$

3. NUMERICAL EXPERIMENTS

The following examples are used to test the efficiency of our new scheme of (2.0.9).

Example 1

$y'' = -y \;\; y(0) = 1\,, y'(o)=1\,, h=0.1$

Theoretical Solution is $y(x) = cosx + sinx$

Example 2

$y'' = x(y')^2 \;\; y(0) = 1\,, y'(o)=0.5\,, h=0.1$

Theoretical Solution is $y(x) = 1 + \frac{1}{2}In\left(\frac{2+x}{2-x}\right)$

Table 1: Approximate solution of Example 1

x	Theoretical Solution	Numerov Method [7]	Hybrid Method [10]	New Method
0.1	1.094837582	1.094837379	1.094837761	1.094837582
0.2	1.178735909	1.178735501	1.178736267	1.178735910
0.3	1.250856696	1.250856127	1.250857196	1.250856698
0.4	1.310479336	1.310478608	1.310479976	1.310479340
0.5	1.357008101	1.357007268	1.357008835	1.357008103

Table 2: Absolute Error of Example 1

Numerov Method [7]	Hybrid Method [10]	New Method
2.03 E(-7)	1.79 E(-7)	---------
4.08 E(-7)	3.58 E(-7)	1.0 E(-9)
5.69 E(-7)	5.0 E(-7)	2.0 E(-9)
7.28 E(-7)	6.4 E(-7)	4.0 E(-9)
8.33 E(-7)	7.34 E(-7)	2.0 E(-9)

Table 3: Approximate solution and Absolute Error of Example 2

x	Theoretical Solution	New Method	Absolute Error
0.1	1.050041729	1.050041729	--------
0.2	1.100335348	1.100335348	-------
0.3	1.151140436	1.151140436	---------
0.4	1.202732554	1.202732555	1.0 E(-9)
0.5	1.255412812	1.255412813	1.0 E(-9)

4. CONCLUSION

The main success of this new scheme is its simplicity of the method, high level of accuracy when compared with existing methods of [7] and [8]. The new method is direct and more efficient than the traditionally method of reducing the problems into system of first order ODEs. It is self starting, no preliminary calculation required before obtaining the next iterations. The results obtained are numerically stable.

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